Optimization of a Manufacturing Management Model by a First Variational Technique

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A multidimensional dynamic manufacturing management system with unknown production and sales functions is considered. This model consisting of seven state variables and three decision variables is optimized by a first variational technique. Convergence rates of advertisement, sales, inventory, and profit are obtained and illustrated through diagrams.

INTRODUCTION

In this article a manufacturing management model to market a new product is considered. Both production and sales are assumed to be unknown. The production rate is a function of time; the sales rate is a function of the contact rate between the persons with the knowledge and the persons without the knowledge about the product, as well as the advertisement rate to promote the sales of the product. This problem is solved by a first variational technique. The results of inventory, advertisement, sales, and profit are shown through figures.

LITERATURE REVIEW

Realistic representation of dynamic industrial and management systems usually result in models involving the optimization of a nonlinear objective function subject to a set of nonlinear differential equation constraints. Various computational techniques have been proposed for solution of such problems. Some of these techniques are dynamic programming, the generalized Newton-Raphson method combined with the calculus of variations or the maximum principle, and other techniques for solving nonlinear boundary value problems resulting from the application of the classical calculus of variations. Each technique has its own limitations and can be used only for a certain class of problems. The various above methods have been compared in detail in the literature of Bhatti (2000), Hendrix and Toth (2010), Jongen et al (2004), Levy (2009), Ruszczynski (2006), Sarker and Newton (2008), Snyam (2005), and Yang et al (2001).

The purpose of this paper is to illustrate the use of the first variational technique, according to Bazaraa et al (2006), Chong and Zak (2008), Fletcher (2001), Luenberger (2008), Pedregal (2004), Smith (1998), and Struwe (2008), for solving manufacturing management problems with end condition constraints. To test the effectiveness of this technique, a seven dimensional management problem involving production, inventory, advertisement, and sales is solved.

A MANUFACTURING MANAGEMENT MODEL

Both sales and production are assumed to be unknown. The process is shown in Figure 1. A raw material is fed into two reactors in series from which the desired product B is obtained. The raw material is a mixture of A, B, and C. Since B is a new product, both inventory and advertisement are assumed for B. We also assume that both A and C have an unlimited market at a fixed price and that they can be sold as soon as manufactured.

The differential equations representing the process are as follows.

$$V_1 \frac{dx_1}{dt} = q(x_0 - x_1) - V_1 k_{a1} x_1 \tag{1}$$

$$V_1 \frac{dy_1}{dt} = q(y_0 - y_1) - V_1 k_{b1} y_1 + V_1 k_{a1} x_1 \tag{2}$$

$$V_2 \frac{dx_2}{dt} = q(x_1 - x_2) - V_2 k_{a2} x_2 \tag{3}$$

$$V_2 \frac{dy_2}{dt} = q(y_1 - y_2) - V_2 k_{b2} y_2 + V_2 k_{a2} x_2 \tag{4}$$

where

x_1, x_2 = concentrations of A in reactors 1 and 2 respectively.

y_1, y_2 = concentrations of B in reactors 1 and 2 respectively.

v_1, v_2 = volumes of reactors 1 and 2 respectively.

x_0, y_0 = concentrations of A and B in the raw material.

$k_{a,1}$, $k_{a,2}$ = reaction rate constants for the production of B in reactors 1 and 2 respectively.

$k_{b,1}$, $k_{b,2}$ = reaction rate constants for the decomposition of B in reactors 1 and 2 respectively.

q = flow rate.

The reaction rate constants are given by the expressions

$$k_{a1} = G_a \exp\left(-\frac{E_a}{RT_1}\right) \qquad k_{b1} = G_b \exp\left(-\frac{E_b}{RT_1}\right)$$

$$k_{a2} = G_a \exp\left(-\frac{E_a}{RT_2}\right) \qquad k_{b2} = G_b \exp\left(-\frac{E_b}{RT_2}\right) \tag{5}$$

where G_a, G_b, E_a, and E_b are given constants. R is the gas constant and

T_1 and T_2 are the temperatures in reactors 1 and 2 respectively.
The rate of change of Inventory is represented by

$$\frac{dI(t)}{dt} = qy_2 - C_q Q(t) \tag{6}$$

where C_q represents the number of items bought by each informed person and Q(t) is the number of informed persons at time t. To determine the sales, the diffusion model due to Teichroew (1964) represented by the following equation is used.

$$\frac{dQ(t)}{dt} = \left(C + a(t)\right)Q(t)\left(1 - \frac{Q(t)}{N}\right) \tag{7}$$

where N is the total number of persons in the group and C is the natural contact coefficient. The number of contacts made can be increased through advertising. The problem is to determine the rate of advertising a(t) so that the total net profit over the duration of the process is maximized.

If I_m represents the optimal inventory level and T_{1m} the feed temperature, the profit equation can be written as

$$J = \int_0^T (C_1 C_q Q(t) + C_2 q x_2 + C_3 q(1 - x_2 - y_2)$$

$$- C_I\left(I_m - I(t)\right)^2 - C_A\left(a(t)Q(t)\right)^2$$

$$- C_T(\left(T_{1m} - T_1\right)^2 + \left(T_1 - T_2\right)^2))dt \tag{8}$$

where C_1, C_2, and C_3 are the revenues per unit from sales of products B, A, and C respectively, C_1 is the cost of inventory, C_A is the cost of advertising, and C_T is the cost of production. Introducing an additional state variable x(t), Equation (8) can be represented by

$$\frac{dx}{dt} = C_1 C_q Q(t) + C_2 q x_2 + C_3 q(1 - x_2 - y_2) - C_I(I_m - I(t))^2$$

$$- C_A\left(a(t)Q(t)\right)^2 - C_T\left(\left(T_{1m} - T_1\right)^2 + \left(T_1 - T_2\right)^2\right) \tag{9}$$

with the initial condition

$$x(0) = 0 \tag{10}$$

The problem now reduces to finding the decision sequences $T_1(t)$, $T_2(t)$, and a(t) so that the final value of x, x(T), is maximized subject to the constraints of Equations (1) through (4) and Equations (6), (7) and (9). This is a seven dimensional state variable problem with three decision variables.

OPTIMIZATION BY A FIRST VARIATIONAL TECHNIQUE

Applying a first variational technique from Bazaraa (2006), Chong (2008), Fletcher (2001), Luenberger (2008), Pedregal (2004), Smith (1998), and Struwe (2008), the following recurrence relations are obtained.

$$\frac{\partial S_1}{\partial x_1}\Big|_t = \frac{\partial S_1}{\partial x_1}\Big|_{t+\Delta} + \left(\frac{\partial S_1}{\partial x_1}\Big|_{t+\Delta}\left(-\frac{q}{V_1} - k_{a1}\right)\right)\Big|_t + \frac{\partial S_1}{\partial y_1}\Big|_{t+\Delta} k_{a1}\Big|_t + \frac{\partial S_1}{\partial x_2}\Big|_{t+\Delta}\left(\frac{q}{V_2}\right)\Big|_t)\Delta \tag{11}$$

$$\frac{\partial S_1}{\partial y_1}\Big|_t = \frac{\partial S_1}{\partial y_1}\Big|_{t+\Delta} + \left(\frac{\partial S_1}{\partial y_1}\Big|_{t+\Delta} \left(-\frac{q}{V_1} - k_{b1} \right)\Big|_t + \frac{\partial S_1}{\partial y_2}\Big|_{t+\Delta} \left(\frac{q}{V_2} \right)\Big|_t \right) \tag{12}$$

$$\frac{\partial S_1}{\partial x_2}\Big|_t = \frac{\partial S_1}{\partial x_2}\Big|_{t+\Delta} + \left(\frac{\partial S_1}{\partial x_2}\Big|_{t+\Delta} \left(-\frac{q}{V_2} - k_{a2} \right)\Big|_t + \frac{\partial S_1}{\partial y_2}\Big|_{t+\Delta} k_{a2}\Big|_t + \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} \left(C_2 q - C_3 q \right)\Big|_t \right)\Delta \tag{13}$$

$$\frac{\partial S_1}{\partial y_2}\Big|_t = \frac{\partial S_1}{\partial y_2}\Big|_{t+\Delta} + \left(\frac{\partial S_1}{\partial y_2}\Big|_{t+\Delta} \left(-\frac{q}{V_2} - k_{b2} \right)\Big|_t + \frac{\partial S_1}{\partial I}\Big|_{t+\Delta} q + \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} \left(-C_3 q \right) \right)\Delta \tag{14}$$

$$\frac{\partial S_1}{\partial I}\Big|_t = \frac{\partial S_1}{\partial I}\Big|_{t+\Delta} + \left(\frac{\partial S_1}{\partial x}\Big|_{t+\Delta} \left(2C_I (I_m - I(t)) \right) \right)\Delta \tag{15}$$

$$\frac{\partial S_1}{\partial Q}\Big|_t = \frac{\partial S_1}{\partial Q}\Big|_{t+\Delta} + (-\frac{\partial S_1}{\partial I}\Big|_{t+\Delta} C_q + \frac{\partial S_1}{\partial Q}\Big|_{t+\Delta} (C + a(t)) \left(1 - \frac{2Q(t)}{N} \right)$$

$$+ \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} \left(C_1 C_q - 2C_a a^2(t) Q(t) \right)\Delta \tag{16}$$

$$\frac{\partial S_1}{\partial x}\Big|_t = \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} \tag{17}$$

The final conditions for Equations (11) through (17) are

$$\frac{\partial S_1}{\partial x_1}\Big|_T = 0\,; \frac{\partial S_1}{\partial y_1}\Big|_T = 0\,; \frac{\partial S_1}{\partial x_2}\Big|_T = 0\,; \frac{\partial S_1}{\partial y_2}\Big|_T = 0\,; \frac{\partial S_1}{\partial I}\Big|_T = 0\,; \frac{\partial S_1}{\partial Q}\Big|_T = 0\,; \frac{\partial S_1}{\partial x}\Big|_T = 1 \tag{18}$$

The recurrence relationships for the decision variables are as follows.

$$\frac{\partial S_1}{\partial T_1}\Big|_t = (\frac{\partial S_1}{\partial x_1}\Big|_{t+\Delta} \left(-x_1 \frac{\partial k_{a1}}{\partial T_1} \right)\Big|_t + \frac{\partial S_1}{\partial y_1}\Big|_{t+\Delta} \left(x_1 \frac{dk_{a1}}{dT_1} - y_1 \frac{dk_{b1}}{dT_1} \right)\Big|_t$$

$$+ \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} (-C_T(-2(T_{1m} - T_1)\big|_t + 2(T_1 - T_2)\big|_t)))\Delta \tag{19}$$

$$\frac{\partial S_1}{\partial T_2}\Big|_t = (\frac{\partial S_1}{\partial x_2}\Big|_{t+\Delta} \left(-x_2 \frac{dk_{a2}}{dT_2} \right)\Big|_t + \frac{\partial S_1}{\partial y_2}\Big|_{t+\Delta} \left(x_2 \frac{dk_{a2}}{dT_2} - y_2 \frac{dk_{b2}}{dT_2} \right)\Big|_t$$

$$+ \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} (-C_T(-2(T_1 - T_2)\big|_t)))\Delta \tag{20}$$

$$\frac{\partial S_1}{\partial a}\Big|_t = \left(\frac{\partial S_1}{\partial Q}\Big|_{t+\Delta} \left(Q(t) - \frac{Q^2(t)}{N} \right) + \frac{\partial S_1}{\partial x}\Big|_{t+\Delta} - \left(2C_a a(t) Q^2(t) \right) \right)\Delta \tag{21}$$

The improved decision can be obtained from

$$\theta_{j\,new}(t) = \theta_{j\,old}(t) + \frac{\Delta\phi_j \frac{\partial S_1}{\partial \theta_j}\Big|_t}{\sum_{t=0}^{T} \frac{\partial S_1}{\partial \theta_j}\Big|_t} \qquad j = 1, 2, 3 \tag{22}$$

where $\Delta\phi_j$ is the desired improvement in the objective function due to the j^{th} decision. Equations (11) through (21) are the recurrence relations for this problem. It should be noted that since the decision vector is multi-dimensional, individual improvements $\Delta\phi_j$ are used for each control.

The numerical values used are listed in Table 1. The initial conditions are

$$x_1(0) = x_2(0) = 0.53$$

$$y_1(0) = y_2(0) = 0.43$$

$$I(0) = 8.0$$

$$Q(0) = 1.0 \tag{23}$$

To start the recursive process, the following initial approximations are used for the decision variables

$$T_1(t) = T_2(t) = 345^{\circ}k; \quad a(t) = 3.0; \text{ for } 0 \le t \le 1 \tag{24}$$

The convergence rates of advertisement, sales, inventory, and profit are shown in Figures 2 through 5. A maximum profit of 66.05 was obtained in approximately 260 iterations.

CONCLUSIONS

The manufacturing management model considered in this paper has previously been solved by a different approach namely a conjugate gradient method by Naadimuthu et al (2009). The first variational method utilized in this work is conceptually much simpler and computationally much easier compared to the conjugate gradient technique. Both the above methods reached approximately the same optimum value of the objective function.

Although in this study we have limited ourselves in the application of the first variational technique to the optimization of a manufacturing management model, this approach can be extended to various other optimization processes dealing with complex business modeling situations.

TABLE 1
NUMERICAL VALUES USED FOR THE PARAMETERS

PARAMETERS	VALUES
R	2.0
q	60.0
V_1	12.0
G_a	0.535×10^{11}
E_a	18000.0
V_2	12.0
G_b	0.461×10^{18}
E_b	30000.0
C_q	1.0
C	1.0
N	100.0
C_1	5.0
C_2	0.0
C_3	0.0
C_I	1.0
C_A	0.01
C_T	0.0005
x_0	0.53
y_0	0.43
T_{1m}	340.0°k
I_m	20.0

FIGURE 1
BLOCK DIAGRAM OF THE MODEL

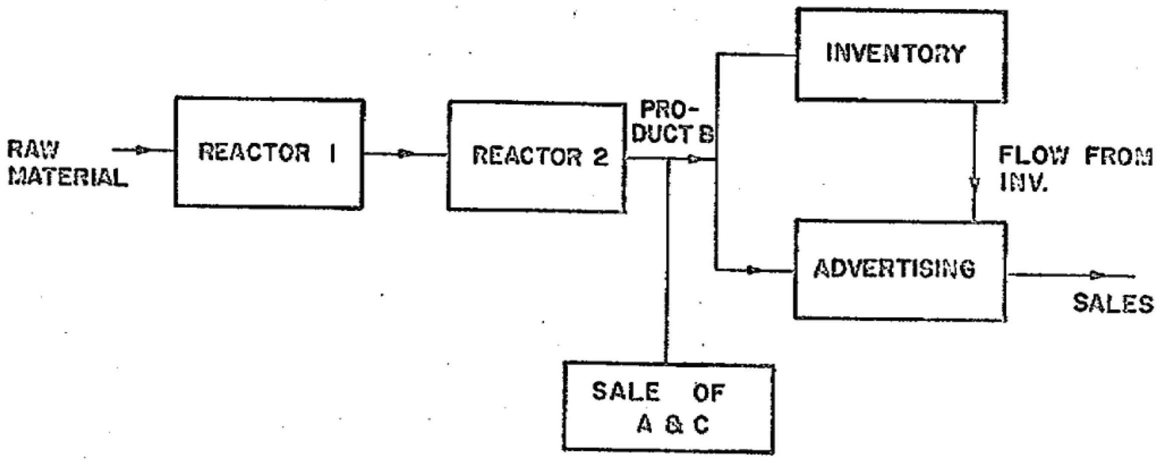


FIGURE 2
CONVERGENCE RATE OF ADVERTISEMENT, a(t)

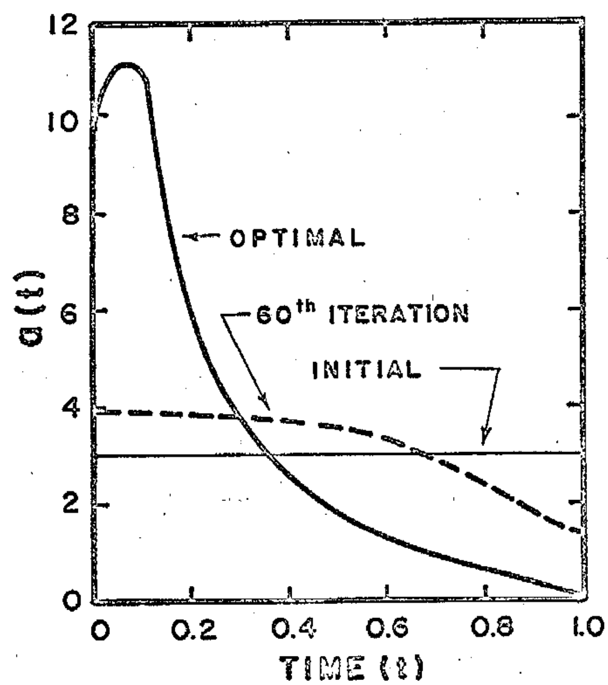


FIGURE 3
CONVERGENCE RATE OF SALES, Q(t)

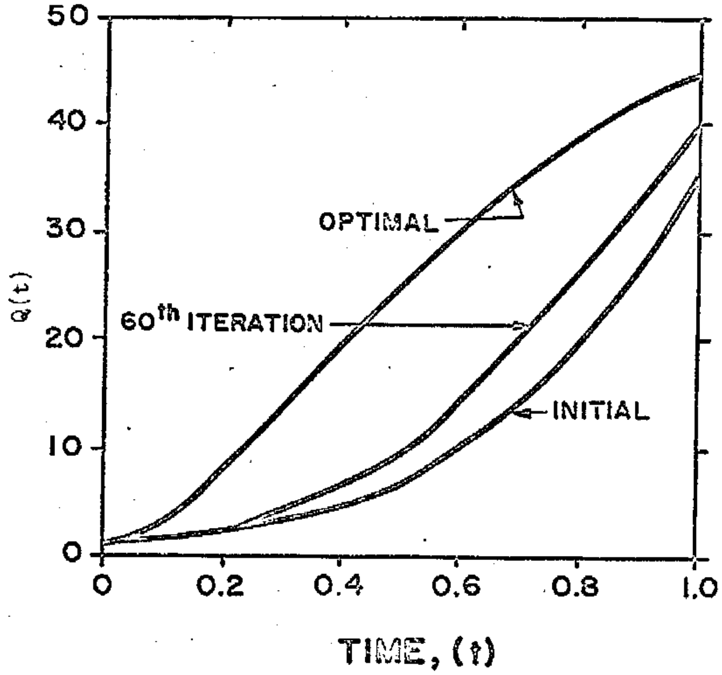


FIGURE 4
CONVERGENCE RATE OF INVENTORY, I(t)

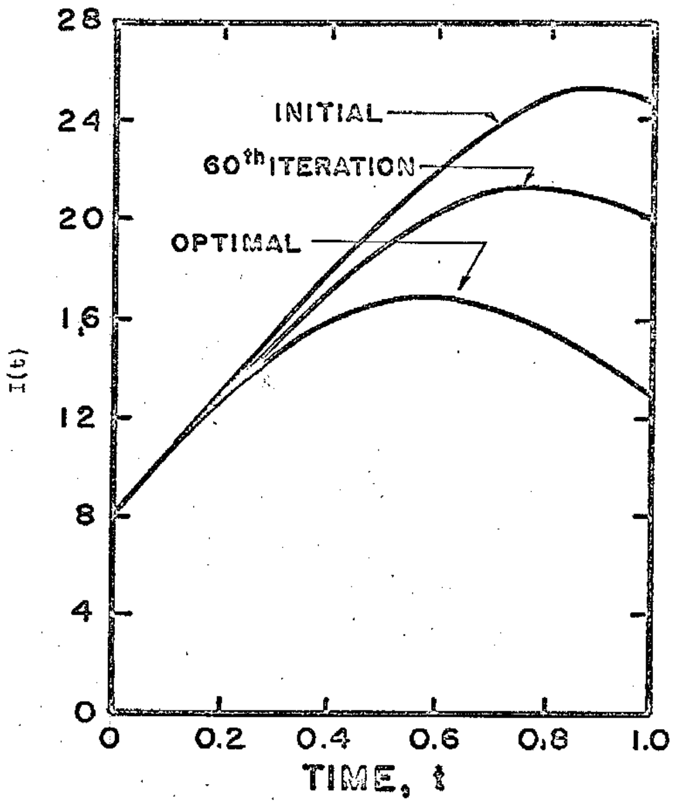
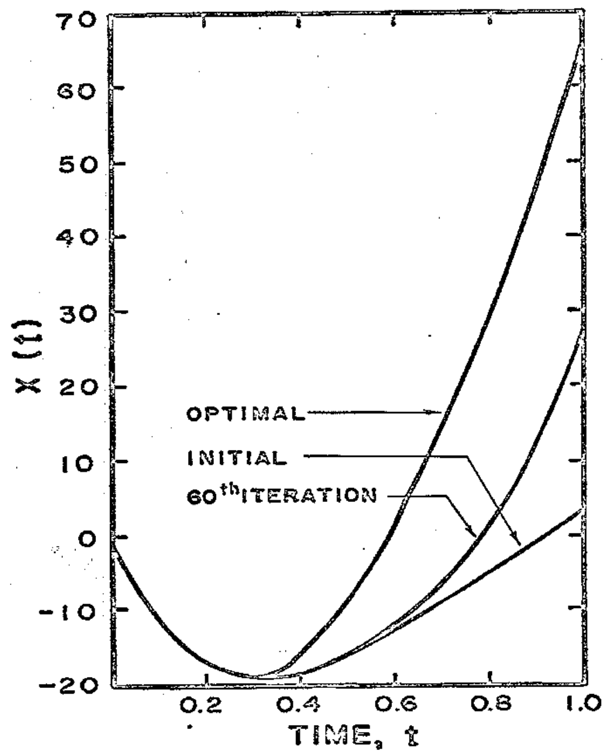


FIGURE 5
CONVERGENCE RATE OF PROFIT, X(t)



REFERENCES

Bazaraa, M.S., Sherali, H.D., and Shetty, C.M. (2006). *Nonlinear Programming: Theory and Algorithms.* Wiley.

Bhatti, M. A. (2000). *Practical Optimization Methods.* Springer.

Chong, K.P., and Zak, S. H. (2008). *An Introduction to Optimization.* Wiley.

Fletcher, R. (2001). *Practical Methods of Optimization.* Wiley.

Hendrix, E.M.T and Toth, B.G. (2010). *Introduction to Nonlinear and Global Optimization.* Springer.

Jongen, H.T., Meer, K., and Triesch, E. (2004). *Optimization Theory.* Kluwer.

Levy, A. B. (2009). *The Basics of Practical Optimization.* SIAM.

Luenberger, D. G. (2008). *Linear and Nonlinear Programming.* Springer.

Naadimuthu, G., Waziruddin, S., and Lee, E.S. (2009). Modeling and Optimization of a Production, Advertisement and Inventory Control System. *Proceedings of the Thirty-Eighth Annual Meeting, Northeast Decision Sciences Institute,* 459-464.

Pedregal, P. (2004). *Introduction to Optimization.* Springer.

Ruszczynski, A. (2006). *Nonlinear Optimization.* Princeton University Press.

Sarker, R. A. and Newton, C.S. (2008). *Optimization Modeling: A Practical Approach.* CRC Press.

Smith, D. R. (1998). *Variational Methods in Optimization.* Dover.

Snyam, J. A. (2005). *Practical Mathematical Optimization.* Springer.

Struwe, M. (2008). *Variational Methods.* Springer.

Teichroew, D. (1964). *An Introduction to Management Science - Deterministic Models.* Wiley.

Yang, X., Teo, K.L., and Caccetta, L. (2001). *Optimization Methods and Applications.* Kluwer.