American Put Option Price Using Quadratic Approximations of the Fractional Black-Scholes Partial Differential Equation

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This study employs fractional Brownian motion (fBM) in modeling the option pricing process and derives two approximation models for American put option valuations. We examine the accuracy of two approximated solutions, denoted by FMBAW and FMQuad, to the fractional Black-Sholes partial differential equation by following the approaches of MBAW and MQuad, respectively. We also empirically compare price errors using put options of petroleum industry - ConocoPhillips, Chevron Corporation, Exxon-Mobil - traded from January, 2005 to December, 2009. These models based on fBm are more accurate and reliable than the corresponding models.

INTRODUCTION

Until recently, option valuation studies conveniently assumed that all information is contained within the current asset price, and thus reasonably considered a Markovian process. However, long-term memory (time dependence) was exhibited by option traders consistently outperforming the market. This triggered many academic studies presuming the existence of a non-Markovian process and the development of stochastic volatility models with quasi long-range dependence (Lo and MacKinlay, 1999). Fractional Brownian motion (fBM) is used to deal with the non-Markovian process mitigating estimation problems of high-dimensional partial differential equations (PDE) with variable coefficients, while still assuming a Gaussian process. fBM offers simple and tractable solutions to financial option valuation and provides a practical way of modeling a non-Markovian price generating process.[1] In a different vein, various studies recognize incentives for early exercise of American options and estimate these early exercise premiums. This incentive is more strongly pronounced for put options because of the timing of cash flows with early exercise. Macmillian (1986) and Barone-Adesi and Whaley (1987) (hereafter, MBAW) initially developed a quadratic approximation model for American put option valuation with early exercise premium. Later Ju and Zhong (1999) (hereafter, MQuad) modified MBAW model and reported an improved result of American option valuation.[2] However, both MBAW and MQuad models are based on Brownian motion, thus they cannot resolve problems with non-Markovian process.

The present study employs fractional Brownian motion (fBM) in modeling the option pricing process and derives two approximation models for American put option valuations with non-Markovian process. This study contributes to the option valuation literature by introducing two non-Markovian valuation models with Gaussian valuation processes.

Elliott and Van der Hoek (2003) derive a European pricing model called the fractional Black-Scholes (FBS) model by utilizing fractional Brownian motion. Heo et al. (2009) show that the FBS model improves the pricing error in European options as compared to the performance of the Black-Scholes (B-S) model using NASDAQ (NDX) index call options. Meng and Wang (2010) claim in foreign exchange option markets that the FBS model performs better than the B-S model. Since the FBS model improves the pricing error in comparison to the B-S model, we adopt the fBM in an approximation of American put option price.

Necula (2007) derives the fractional Black-Sholes PDE replacing the Black-Scholes equation, where the FBS model (2003) satisfies the fractional Black-Scholes PDE. Because American put option value is the sum of European put and early exercise premium, this fractional PDE offers an American option pricing model with moving boundary conditions as established by McKean (1965).[3] Unfortunately the Necula's fractional Black-Sholes PDE appears not to have an exact solution in estimation of American put early exercise premium. Heo et al. (2010) estimates American put option values with the FBS model replacing B-S to value European put options and to identify critical stock price to estimate the early exercise premium. Therefore, they are referred as a hybrid of two methods – MBAW and MQuad. In this study, our two valuation models provide approximations to the fractional Black-Sholes PDE by modifying the approaches of MBAW and MQuad. This study is different from Heo et al. (2010) because the new models estimate an early exercise premium on American put option from the fractional Black-Scholes PDE. Because no previous study used the fractional PDE in evaluating American option models, findings from this study will have practical implications in risk hedging and financial engineering.

We examine the accuracy of American put valuation models base on fractional Brownian motion using equity option data of petroleum industry – Conoco Phillips (COP), Chevron Corporation (CVX), Exxon Mobil (XOM) – traded on the Chicago Board of Trade Option Exchange (CBOE) from January 1, 2005 to December 31, 2009. The accuracy is measured with mean absolute percentage error with respect to option price (MAPE), mean percent error with respect to option price (MPE) and root mean squared error (RMSE) by moneyness, volatilities, and option maturity.

The remainder of this paper is organized as follows: Section II derives quadratic approximation American put option models using fractional Black Scholes PDE. In Section III, we describe the methodology and data screen process. Section IV discusses the empirical findings. The final section concludes and suggests for future research.

MODELS

Fractional Black-Scholes Partial Differential Equation

Using the time variable t, $0 \le t \le T$, where $t = 0$ corresponds to the issue date of the option and $t = T$ corresponds to its expiration date, we let $S = S(t)$ = stock price at time t, X = strike price of option r = current risk-free interest rate, σ= stock price volatility, $\tau = (T - t)$ = time to expiration, and δ = current dividend yield (for dividend paying stocks). Then European put option price, $P_E(S,t)$, is given by

$$P_E(S,t) = Xe^{-r\tau}N(-d_2) - Se^{-\delta\tau}N(-d_1), \tag{1}$$

where $d_1 = \frac{\ln\left(\frac{S}{X}\right)+(r-\delta)\tau+\frac{\sigma^2}{2}(T^{2H}-t^{2H})}{\sigma\sqrt{T^{2H}-t^{2H}}}$, $d_2 = d_1 - \sigma\sqrt{T^{2H}-t^{2H}}$, and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du$.

Let $V(S,t)$ be the price of an option on a stock where S is the price of the stock at time t. Then $V(S,t)$ satisfies the partial differential equation

$$H\sigma^2\, t^{2H-1} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t} = 0, \tag{2}$$

which is called the fractional Black-Scholes PDE (Necula, 2007). The classical Black-Scholes PDE is the special case of the above equation if $H = 1/2$.

Quadratic Approximation Solutions to Fractional Black-Scholes PED

In this section, we derive two quadratic approximation formulas, FMBAW and FMQuad, by following the approaches of MBAW and MQuad, respectively, using the fractional Black-Scholes differential equation.

FMBAW Method

If an American put option price satisfies $P = P_E + \rho$ where ρ is the early exercise premium and P_E is the European put value, we have the following boundary conditions in term of ρ: $\rho(S,T) = 0$ and $\lim_{S\to\infty} \rho(S,t) = 0$. At the critical stock price S^*, we have two moving boundary conditions

$$\lim_{S\to S^*} \rho(S,t) = X - S^* - P_E(S,t), \tag{3}$$

$$\lim_{S\to S^*} \frac{\partial \rho(S,t)}{\partial S} = -1 - \frac{\partial\, P_E(S^*,t)}{\partial S}. \tag{4}$$

For values of $S > S^*$, the early exercise premium ρ satisfies the fractional Black-Scholes differential equation

$$H\sigma^2\, t^{2H-1} S^2 \frac{\partial^2 \rho}{\partial S^2} + (r-\delta) S \frac{\partial \rho}{\partial S} - r\rho + \frac{\partial \rho}{\partial t} = 0. \tag{5}$$

The exact solution to (5) is unknown so an approximation solution to the equation can derived by following the same approach of MacMillan (1986), Barone-Adesi, and Whaley (1987) (MBAW)
For simplicity, we let $\tau = (T-t)$, $M_1 = r/\sigma^2$, $M_2 = (r-\delta)/\sigma^2$, and $L(\tau) = H\,(T-\tau)^{2H-1}$. Then, equation (5) becomes

$$L(\tau)\, S^2 \frac{\partial^2 \rho}{\partial S^2} + M_2 S \frac{\partial \rho}{\partial S} - M_1 \rho - \frac{M_1}{r} \frac{\partial \rho}{\partial \tau} = 0. \tag{6}$$

As in the MBAW method, let's assume that $\rho = K(\tau)\Psi(S,K)$. Then $\Psi(S,K)$ satisfies

$$L(\tau) K(\tau) S^2 \frac{\partial^2 \Psi}{\partial S^2} + M_2 K(\tau) S \frac{\partial \Psi}{\partial S} - M_1 K(\tau)\Psi - \frac{M_1}{r}\left[\frac{dK}{d\tau}\Psi + K(\tau)\frac{dK}{d\tau}\cdot\frac{\partial \Psi}{\partial K}\right] = 0. \tag{7}$$

By combining the last two terms in (7) and assuming $\frac{dK}{d\tau} = r\big(1 - K(\tau)\big)$, thus $K(\tau) = 1 - e^{-r\tau}$, and equation (7) becomes

$$L(\tau) S^2 \frac{\partial^2 \Psi}{\partial S^2} + M_2 S \frac{\partial \Psi}{\partial S} - \frac{M_1}{K(\tau)} \Psi \left[1 + \big(1 - K(\tau)\big)\frac{K(\tau)}{\Psi}\cdot\frac{\partial \Psi}{\partial K}\right] = 0. \tag{8}$$

Further simplification leads to

$$L\,S^2\frac{\partial^2\Psi}{\partial S^2}+M_2S\frac{\partial\Psi}{\partial S}-\frac{M_1}{K}\Psi-(1-K)M_1\frac{\partial\Psi}{\partial K}=0. \quad (9)$$

As $\tau\to 0$ or $\tau\to\infty$, the last term $(1-K)M_1\frac{\partial\Psi}{\partial K}\to 0$. Hence it is reasonable to assume the last term in (9) is zero. Let $\alpha=M_1/L$ and $\beta=M_2/L$. Then we obtain

$$S^2\frac{\partial^2\Psi}{\partial S^2}+\beta S\frac{\partial\Psi}{\partial S}-\frac{\alpha}{K}\Psi=0. \quad (10)$$

The solution to (10) provides the approximation to the fractional Black-Scholes PDE (5). Because equation (10) is a second-order ordinary differential equation, it has two linearly independent solutions of the form$\Psi=a\,S^{\lambda}$, where is λ is the solution of the equation $a\,S^{\lambda}\left[\lambda^2+(\beta-1)\lambda-\frac{\alpha}{K}\right]=0.$ Hence,

$$\lambda=\frac{(1-\beta)\ \pm\sqrt{(1-\beta)^2+4\alpha/K}}{2}. \quad (11)$$

Let

$$\lambda_1=\frac{(1-\beta)-\sqrt{(1-\beta)^2+4\alpha/K}}{2}, \text{ and } \lambda_2=\frac{(1-\beta)+\sqrt{(1-\beta)^2+4\alpha/K}}{2}. \quad (12)$$

Then $\Psi=a_1\,S^{\lambda_1}+a_2\,S^{\lambda_2}$. Since $\lambda_2>0$, as $S\to\infty,\Psi\ \to\infty$. Therefore a_2 should be zero. Otherwise, the early exercise premium ρ approaches infinity. Hence $\Psi=a_1\,S^{\lambda_1}$ and

$$P(S,t)=P_E(S,t)+K\,a_1S^{\lambda_1}. \quad (13)$$

The values of a_1 and the critical stock price $S^*$are determined by imposing the boundary conditions (3) and (4). From boundary condition (4) and the continuity of the derivative, we have

$$-1=\frac{\partial\,P_E(S^*,t)}{\partial S}+\lambda_1Ka_1{S^*}^{\lambda_1-1}. \quad (14)$$

As a result of the continuity of option price at boundary condition (3),

$$X-S^*=P_E(S^*,t)+K\,a_1{S^*}^{\lambda_1}. \quad (15)$$

Because $\frac{\partial\,P_E(S^*,t)}{\partial S}=-e^{-\delta\tau}N(-d_1(S^*))$, then $a_1=\frac{-1+e^{-\delta\tau}N(-d_1(S^*))}{\lambda_1K{S^*}^{\lambda_1-1}}$, which substituted into (15), we obtain the critical stock price S^* by solving

$$X-S^*=P_E(S^*,t)-\frac{1}{\lambda_1}\left[1-e^{-\delta\tau}N\big(-d_1(S^*)\big)\right]S^*. \quad (16)$$

The solution for the American put option price is then easily obtained as follows.

$$P(S,t) = \begin{cases} P_E(S,t) + KA(K)\left(\frac{S}{S^*}\right)^{\lambda(K)} & if\ S > S^* \\ X - S & if\ \ S \leq S^*, \end{cases} \tag{17}$$

where $A = -\left(\frac{S^*}{\lambda K}\right)\left[1 - e^{-\delta\tau}N\left(-d_1(S^*)\right)\right]$ and $\lambda(=\lambda_1) = \frac{(1-\beta)-\sqrt{(1-\beta)^2+4\alpha/K}}{2}$.

We note here that the put price value formula $P(S,t)$ in (17) is exactly the same as the price formula in Barone-Adesi, and Whaley (1987) (MBAW) when $H = \frac{1}{2},\ M = \alpha,$ and $N = \beta$.

FMQuad Method

Ju and Zhong (1999) modified the assumptions of MBAW by assuming ρ is of the form $K(\tau)\left(\Psi_1(S,K) + \Psi_2(S,K)\right)$, where Ψ_1 is the same Ψ above in the MBAW method, and Ψ_2 represents an additional correction term. Since the MBAW method captures most of the early exercise premium, Ψ_2 is relatively small so that it is assumed that $\Psi_2 = \varepsilon\Psi_1$with a small ε. Then equation (9) becomes

$$S^2\frac{\partial^2\Psi_1}{\partial S^2} + \beta S\frac{\partial\Psi_1}{\partial S} - \frac{\alpha\Psi_1}{K} + S^2\frac{\partial^2\,\varepsilon\Psi_1}{\partial S^2} + \beta S\frac{\partial\varepsilon\Psi_1}{\partial S} - \frac{\alpha\varepsilon\Psi_1}{K} - (1-K)M_1\frac{\partial(\Psi_1 + \varepsilon\Psi_2)}{\partial K} = 0 \tag{18}$$

With $\Psi_1 = A(S/S^*)^\lambda$ satisfying equation (10), the first three terms in (18) are zero. Hence equation (18) simplifies to

$$S^2\frac{\partial^2\,\varepsilon\Psi_1}{\partial S^2} + \beta S\frac{\partial\varepsilon\Psi_1}{\partial S} - \frac{\alpha\varepsilon\Psi_1}{K} - (1-K)M_1\frac{\partial(\Psi_1 + \varepsilon\Psi_2)}{\partial K} = 0. \tag{19}$$

Since $\frac{\partial\varepsilon\Psi_1}{\partial S} = \frac{\partial\varepsilon}{\partial S}\Psi_1 + \varepsilon\frac{\partial\Psi_1}{\partial S}$ and $\frac{\partial^2\,\varepsilon\Psi_1}{\partial S^2} = \frac{\partial^2\,\varepsilon}{\partial S^2}\Psi_1 + 2\frac{\partial\varepsilon}{\partial S}\frac{\partial\Psi_1}{\partial S} + \varepsilon\frac{\partial^2\,\Psi_1}{\partial S^2}$, dividing (19) by Ψ_1, we obtain

$$S^2\frac{\partial^2\,\varepsilon}{\partial S^2} + \left(\frac{2S^2}{\Psi_1}\frac{\partial\Psi_1}{\partial S} + \beta S\right)\frac{\partial\varepsilon}{\partial S} - (1-K)\alpha\left[\frac{1+\varepsilon}{\Psi_1}\frac{\partial\Psi_1}{\partial K} + \frac{\partial\varepsilon}{\partial K}\right] = 0. \tag{20}$$

As in Ju and Zhong, we assume that $\frac{\partial\varepsilon}{\partial K} = 0$. From $\Psi_1 = A(K)(S/S^*)^\lambda$, $\frac{\partial\Psi_1}{\partial S} = \Psi_1\frac{\lambda}{S}$ and $\frac{\partial\Psi_1}{\partial K} = \Psi_1\left[\frac{A'(K)}{A(K)} + \lambda'\log\left(\frac{S}{S^*}\right) - \frac{\lambda}{S^*}\frac{\partial S^*}{\partial K}\right]$. Hence from (20), we have

$$S^2\frac{\partial^2\,\varepsilon}{\partial S^2} + (2\lambda+\beta)S\frac{\partial\varepsilon}{\partial S} - (1-K)\alpha(1+\varepsilon)\left[\frac{A'(K)}{A(K)} + \lambda'\log\left(\frac{S}{S^*}\right) - \frac{\lambda}{S^*}\frac{\partial S^*}{\partial K}\right] = 0. \tag{21}$$

To solve the second order ODE (21), $(1+\varepsilon)$ is treated as a constant. Then the solution to equation (21) can be written as $\varepsilon = y_p + y_h$, where $y_h = c_1\frac{S^{-(2\lambda+\beta-1)}}{2\lambda+\beta-1} + c_2$ and $y_p = B(K)Y^2 + C(K)Y$, where $Y = \log\left(\frac{S}{S^*}\right)$. Because $2\lambda + \beta - 1$ is negative, y_h approaches ∞ as S approaches ∞. Therefore, c_1 and c_2 must be zero. Hence ε takes the form $B(K)Y^2 + C(K)$, where $B(K) = \frac{(1-K)\lambda'(K)\alpha(1+\varepsilon)}{2(2\lambda+\beta-1)}$ and Hence ε takes the form $B(K)Y^2 + C(K)$, where $B(K) = \frac{(1-K)\lambda'(K)\alpha(1+\varepsilon)}{2(2\lambda+\beta-1)}$ and $C(K) = \frac{(1-K)\alpha(1+\varepsilon)}{(2\lambda+\beta-1)}\left[\frac{A'(K)}{A(K)} - \frac{\lambda'(K)}{(2\lambda+\beta-1)} - \frac{\lambda}{S^*}\frac{\partial S^*}{\partial K}\right]$, where $\lambda'(K) = \frac{\alpha}{K^2\sqrt{(1-\beta)^2+\frac{4\alpha}{K}}}$.

Let $b = \frac{(1-K)\lambda'(K)\alpha}{2(2\lambda+\beta-1)}$, and $c = \frac{(1-K)\alpha}{(2\lambda+\beta-1)}\left[\frac{A'(K)}{A(K)} - \frac{\lambda'(K)}{(2\lambda+\beta-1)} - \frac{\lambda}{S^*}\frac{\partial S^*}{\partial K}\right]$. Then from boundary condition (3), as S approaches S^*, we obtain $X - S^* = P_E(S^*,t) + KA(K)$. By differentiating with respect to K,

$$A'(K) = -\frac{1}{K}\left[\left(1 - e^{-\delta\tau}N\left(-d_1(S^*)\right)\right)\frac{\partial S^*}{\partial K} + \frac{\partial P_E(S^*,\tau)}{\partial K} + A(K)\right]. \tag{22}$$

Hence,

$$c = \frac{-(1-K)\alpha}{(2\lambda+\beta-1)}\left[\left(\frac{1-e^{-\delta\tau}N\left(-d_1(S^*)\right)}{KA(K)} - \frac{\lambda(K)}{S^*}\right)\frac{\partial S^*}{\partial K} + \left(\frac{1}{KA(K)}\frac{\partial P_E(S^*,\tau)}{\partial K} + \frac{1}{K} + \frac{\lambda'(K)}{2\lambda+\beta-1}\right)\right], \tag{23}$$

where $\frac{\partial P_E(S^*,\tau)}{\partial K} = \frac{\partial P_E(S^*,\tau)}{\partial \tau}\frac{\partial \tau}{\partial K} = -XN(-d_2) + \frac{S^*\delta}{r}e^{(r-\delta)\tau}N(-d_1) + \frac{X}{r}\frac{\partial N(-d_2)}{\partial \tau} - \frac{s}{r}e^{(r-\delta)\tau}\frac{\partial N(-d_1)}{\partial \tau}$.

As in MBAW method, the critical stock price S^* is recovered from the equation

$$X - S^* = P_E(S^*,t) - \frac{1}{\lambda}\left[1 - e^{-\delta\tau}N\left(-d_1(S^*)\right)\right]S^*, \tag{24}$$

which causes the term involving $\frac{\partial S^*}{\partial K}$ in (23) to equal zero. As a result,

$$c = -\frac{(1-K)\alpha}{(2\lambda+\beta-1)}\left[\frac{1}{KA(K)}\frac{\partial P_E(S^*,\tau)}{\partial K} + \frac{1}{K} + \frac{\lambda'(K)}{2\lambda+\beta-1}\right]. \tag{25}$$

Then the price of an American option is approximated by

$$P(S,t) = \begin{cases} P_E(S,t) + \dfrac{KA(K)(S/S^*)^{\lambda(K)}}{1-\chi} & if \quad S > S^* \\ X - S & if \quad S \le S^*, \end{cases} \tag{26}$$

where $\chi = b\left[\log\left(\frac{S}{S^*}\right)\right]^2 + c\log\left(\frac{S}{S^*}\right)$.

The option pricing formulas (17) and (26) resemble the formulas given by Barone-Adesi, and Whaley (1987) and Ju and Zhong (1999), respectively. However, the coefficients α and β involve the term $L(t) = H \cdot t^{2H-1}$ so that the critical stock prices and early exercise premium are very sensitive to H values which separate the Black-Scholes PDE and fractional Black-Scholes PDE unless $H = 1/2$.

DATA AND METHODOLOGY

This study uses daily market closing prices traded on the Chicago Board of Option Exchange (CBOE) for put options of three petroleum companies: ConocoPhillips (COP), Chevron Corporation (CVX), and Exxon Mobil (XOM). The data spans the time period from January 1, 2005 to December 31, 2009. These put options differ in exercise price and expiration date. As a result, there are 1,083 COP, 758 CVX, and 873 XOM equity put options.

It is necessary to filter the data for a variety of reasons. First, American put option prices must satisfy the no-arbitrage boundary conditions, given as:

$$C_{actual} - S + Xe^{-rT} \le P_{model} \le C_{actual} - S + X \quad \text{and} \quad P_{actual} \le 0.99(X - S). \tag{27}$$

Any observation failing these conditions was deleted. Observations with put option prices less than \$0.50 were also deleted to eliminate outliers and prohibitively high transaction costs. Thinly traded put options were also deleted. Some input variables such *H* value and implied volatility were recovered from the previous day's data, thus the first observation of each put option was lost. The last filter required deleting options whose *H* values are not in the range of 0 to 1. The final dataset consists of 74,833 usable observations (26,330 for COP, 20,748 for CVX and 27,755 for XOM).

Standard option definitions based on different moneyness apply. Namely, given the different ranges for moneyness (S/X), different options are defined to be:

Deep-in-the-money (DITM)	S/X < 0.95
In-the-money (ITM)	0.95 < S/X <0.98
At-the-money (ATM)	0.98 < S/X <1.02
Out-of-the-money (OTM)	1.02 < S/X < 1.05
Deep-out-of-the-money (DOTM)	1.05< S/X.

The option maturity is divided into four periods at three month intervals. The first, $\boldsymbol{M_1}$ is defined as a maturity of less than three months, $\boldsymbol{M_2}$ between three and six months, $\boldsymbol{M_3}$ between six months and nine months, and lastly, $\boldsymbol{M_4}$ is a maturity longer than nine months.

The study employs three measures to evaluate the accuracy with respect to option price of each model. These are mean absolute percentage error (*MAPE*), mean percentage error (*MPE*), and root mean squared error (*RMSE*), given by the following equations respectively,

1) $$MAPE = \frac{1}{N}\sum\frac{|P_{model}-P_{actual}|}{P_{actual}} \times 100(\%)$$

2) $$MPE = \frac{1}{N}\sum\frac{P_{model}-P_{actual}}{P_{actual}} \times 100(\%)$$

3) $$RMSE = \sqrt{\frac{1}{N}\sum(P_{model} - P_{actual})^2}\ ,$$

where P_{actual} is the actual put option price; and P_{model} is the model-generated price; and *N* is the number of observations.

This study utilizes two common volatility measures, implied (IV) and historical volatility (HV). The implied volatility measures are recovered from the binomial tree model with 100 steps and historical volatility values are measured by historical standard deviation of log returns for the previous three months. We recover the Hurst values (*H*) from the FBS model which depends on the two volatility measures, IV and HV.

EMPIRICAL RESULTS

This study's accuracy tests results are reported in the following tables. In Table 1, the summary of MAPE, RMSE, and MPE by volatilities are presented. These three accuracy measures, sorted by option maturity and moneyness, are reported in Table 2 and Table 3, respectively. Each table is separated horizontally by two categories, one for implied volatility (IV) and the other for historical volatility (HV), and vertically by Group 1 (B-S, MBAW, MQuad - non-fractional models) and Group2 (FBS, FMBAW, FMQuad – fractional models). Because models using implied volatility yielded better accuracy than using historical volatility, we present MAPE with IV across two dimension, maturity and moneyness in Table 4. The results for RMSE and MPE with IV across maturity and moneyness are included in Appendix. We can conclude from Table 1 that all models using implied volatility (IV) yield more accurate estimations

(MAPE, MPE) and less variance (RMSE) than using historical volatility (HV) for all model specifications consistent with the literature. From Heo et al. (2009), it is expected that the models in Group 2 should perform better than those in Group 1 in estimating option prices because the FBS model improves European option price errors, and also, the Hurst parameter H better captures fluctuations of the market than does Brownian motion. Indeed, as expected all models in Group 2 outperform the corresponding models in Group 1.[4] In particular FMQuad outperforms the other models except for the case of MPE with HV where FMBAW is more accurate. We noticed that American option models outperform the corresponding European option models as we expected. The European FBS option price model yields surprisingly less MAPE and MPE than the American option models in Group 1. The improvement in accuracy between the two groups is more dramatic where HV is concerned. MBAW performs best in Group 1 and FMQuad performs best in Group 2 excepting the MPE case. This finding contradicts, when MBAW and MQuad are compared, the claim by Ju and Zhang (1999), where they use only 87 simulation data entries.

TABLE 1
MAPE, RMSE, AND MPE BY VOLATILITIES

		Group 1			Group 2		
Volatility	**Measure**	**B-S**	**MBAW**	**MQuad**	**FBS**	**FMBAW**	**FMQuad**
IV	**MAPE**	11.1454	**5.8778**	6.3777	4.5603	4.5369	**4.1204**
	RMSE	1.3944	**0.524**	0.5291	0.6576	0.5032	**0.4445**
	MPE	-11.06	**-5.6841**	-6.2055	-3.506	0.8913	**0.3379**
HV	**MAPE**	21.3525	**16.85**	17.1846	7.8045	7.4747	**7.0984**
	RMSE	1.7083	**0.9955**	1.0055	0.7238	0.5467	**0.4964**
	MPE	-20.472	**-15.473**	-15.858	-6.7794	**-2.8421**	-3.3207

As we notice in Table 1, the differences in MAPE and RMSE among the European FBS model and the American models in Group 2 are insignificant. We scrutinize the pricing errors by option maturity (Table 2) and moneyness (Table 3). Their differences between models start to reveal more clearly. If the option maturity is considered, all models in Group 2 outperform the corresponding models in Group 1. Thus we will focus our discussion more to models in Group 2.

American option models in Group 2 generate more accurate option prices except FBS in M_4 case. Particularly, when the maturity is less than 10 months (M_1, M_2, M_3), we find that, regardless of volatilities, FMBAW is more accurate than other models if MAPE is considered and FMQuad yields less error than others if RMSE is considered. This pattern is very similar between MBAW and MQuad.

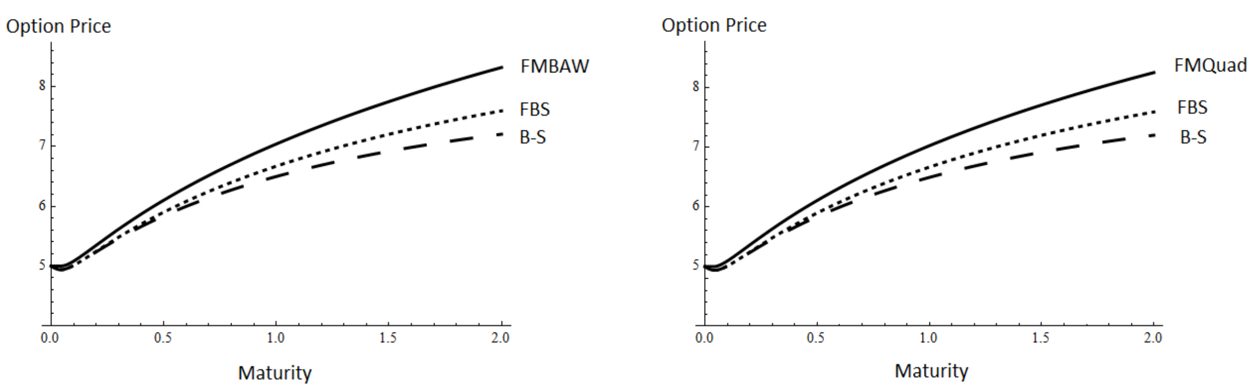
This result is similar to the result in Heo et al. (2010) which used the hybrid models of MBAW and MQuad studying financial option prices. MPE results suggest that all models underestimate option prices except with M_4 cases for FMBAW and FMQuad. We observe the same phenomenon when HV is considered. It is known that the shorter the maturity, the better the accuracy. This is the case except for FBS. If the maturity is longer than 9 months (M_4), FBS significantly outperforms the other models in all three measures, which is surprising because it is a European option model.[6] The FBS (M_4) result is the best among all the maturity periods. This can be partially explained if we exam the MPE in Table 2, where FBS with IV underestimates the actual option price by 0.2294%, where as FMBAW and FMQuad overestimate the option price by 6.0555% and 4.3107%, respectively. Because FBS estimation is already within 0.2294% of the actual price, adding the early exercise premium contributes even bigger approximation error to FMBAW and FMQuad unless the American option models capture early exercise premium exactly.

TABLE 2
MAPE, RMSE, AND MPE BY MATURITY[5]

			Group 1			Group 2		
Measure	**Volatility**	**Maturity**	**B-S**	**MBAW**	**MQuad**	**FBS**	**FMBAW**	**FMQuad**
MAPE	**IV**	M_1	3.4032	2.0839	2.0812	2.585	1.6456	1.6605
		M_2	7.9084	4.4457	4.5417	6.5667	3.7335	3.8449
		M_3	10.5981	6.1434	6.4692	8.8018	4.8567	5.3999
		M_4	17.9027	8.8269	9.9639	2.2483	6.5322	5.0553
	HV	M_1	4.6198	3.3926	3.4021	3.2989	2.3317	2.3684
		M_2	14.9307	11.7149	11.8242	12.2251	9.4958	9.624
		M_3	23.3845	19.3535	19.6201	18.2137	14.134	14.1524
		M_4	34.0164	26.5956	27.2966	2.2827	5.7442	4.606
RMSE	**IV**	M_1	0.4815	0.256	0.2532	0.4556	0.2449	0.2441
		M_2	0.8945	0.3371	0.3245	0.8553	0.3204	0.3156
		M_3	1.0361	0.3995	0.385	0.9783	0.348	0.3358
		M_4	2.0227	0.7452	0.7628	0.3182	0.7254	0.6177
	HV	M_1	0.4963	0.2724	0.2706	0.4685	0.2539	0.2536
		M_2	0.968	0.4411	0.434	0.923	0.4104	0.4086
		M_3	1.2077	0.6463	0.6398	1.1196	0.5529	0.5343
		M_4	2.5397	1.5221	1.5427	0.3436	0.7172	0.6185
MPE	**IV**	M_1	-3.1322	-1.5944	-1.5659	-1.9062	-0.4791	-0.4449
		M_2	-7.846	-4.272	-4.3965	-6.1305	-2.7697	-2.907
		M_3	-10.555	-6.0303	-6.3864	-8.4458	-3.0861	-2.4312
		M_4	-17.885	-8.7451	-9.9205	-0.2294	6.0555	4.3107
	HV	M_1	-3.8166	-2.2782	-2.2494	-2.7078	-1.2678	-1.2324
		M_2	-14.49	-10.988	-11.083	-11.884	-8.5662	-8.6761
		M_3	-22.913	-18.591	-18.862	-17.836	-12.758	-12.339
		M_4	-32.597	-24.335	-25.202	-0.2402	4.9529	3.506

M_1 (Less than 3 months), M_2 (between 3 and 6 months), M_3(between 6 and 9 months), M_4(greater than 9 months)

FIGURE 1
OPTION PRICE INFLUENCED BY MATURITY



In Figure 1, we present the changes of option prices of B-S, FBS, FMBAW and FMQuad assuming that all models have the same parameter values[7] over two year maturity $(T-t)$ periods. As the time to maturity gets larger, so does the option price, which is expected due to the time value. However, the rates of the increase of FMBAW and FMQuad are greater than those of FBS and B-S. Our empirical study shows that the increase is significant when the time to maturity is 9 month and beyond, regardless of the volatilities and moneyness (Table 4). Structural bias of FMBAW and FMQuad is inevitable the longer the maturity as demonstrated in Figure 1.

Table 3 compares pricing errors by moneyness. When IV with MAPE is considered, FMQaud performs best for DITM and ITM cases and FBS outperforms for the other cases. When IV with RMSE is considered, FBS outperforms the other models except in the DITM case where FMQuad performs best. However, the differences between FBS and FMQuad are negligible (0.4111 vs. 0.4117) in ITM case with RMSE. Hence it is safe to say that FMQuad is effective in DITM and ITM cases and FBS is reliable in ATM, OTM and DOTM cases. Because the number of observations in DITM is more than a half of the data set, FMQad is reported as the best model in Table 1. For HV, using MAPE and RMSE measures,

TABLE 3
MAPE, RMSE, AND MPE BY MONEYNESS[8]

			Group 1			Group 2		
Measure	**Volatility**	**Moneyness**	**B-S**	**MBAW**	**MQuad**	**FBS**	**FMBAW**	**FMQuad**
MAPE	**IV**	**DITM**	8.1843	2.8509	2.7459	3.9317	2.8279	2.5466
		ITM	11.5961	6.5913	6.6848	4.8472	5.0241	4.6414
		ATM	12.0728	7.3278	7.6119	4.7157	5.4128	5.0848
		OTM	12.9184	8.332	8.8239	5.195	5.7932	5.4749
		DOTM	15.6187	10.2283	11.8476	5.4522	6.9867	6.3083
	HV	**DITM**	8.6514	3.5122	3.4263	3.9969	2.938	2.6835
		ITM	14.2087	9.7952	9.8585	5.2991	5.5988	5.1909
		ATM	16.2246	12.2224	12.4206	5.696	6.3941	5.9844
		OTM	19.763	15.872	16.2515	7.45	7.7996	7.3226
		DOTM	44.9462	41.3185	42.4204	15.0258	15.5319	14.9765
RMSE	**IV**	**DITM**	1.7869	0.5962	0.5816	0.8751	0.6314	0.5579
		ITM	1.1274	0.6133	0.6345	0.4111	0.4595	0.4117
		ATM	0.9721	0.5458	0.5759	0.3045	0.3985	0.3512
		OTM	0.881	0.5235	0.5606	0.2792	0.3526	0.3129
		DOTM	0.5535	0.3446	0.3855	0.1677	0.2266	0.1979
	HV	**DITM**	1.9602	0.86	0.8476	0.9275	0.6579	0.5893
		ITM	1.6112	1.1308	1.1469	0.4949	0.4978	0.4548
		ATM	1.5183	1.1363	1.1592	0.4137	0.4442	0.4059
		OTM	1.5005	1.1788	1.209	0.4194	0.423	0.393
		DOTM	1.2758	1.1206	1.1528	0.3438	0.3437	0.334
MPE	**IV**	**DITM**	-8.1719	-2.7752	-2.658	-3.5753	0.3967	0.305
		ITM	-11.494	-6.3691	-6.4266	-3.9231	0.2572	0.159
		ATM	-11.804	-6.8998	-7.1749	-3.3817	0.7414	0.4604
		OTM	-12.646	-7.9527	-8.46	-3.7838	0.3211	-0.0493
		DOTM	-15.475	-9.9111	-11.624	-3.3098	1.9262	0.4509
	HV	**DITM**	-8.5632	-3.0955	-2.9752	-3.6288	0.3637	0.2669
		ITM	-13.415	-8.2659	-8.3012	-4.2767	-0.2796	-0.4023
		ATM	-14.508	-9.635	-9.8837	-4.2673	-0.3669	-0.6561
		OTM	-18.052	-13.482	-13.956	-6.0151	-2.2838	-2.6655
		DOTM	-42.995	-38.723	-40.025	-13.032	-9.1613	-10.385

FMQuad dominates with regards to accuracy, except for in the ATM case, where the FBS model yields less error as measured using MAPE. Among models in Group 1, MQuad only outperforms the other two models in the DITM case. MBAW yields smaller errors in all other cases, which is quite different from the results in Group 2 where FMBAW pricing errors lie between FBS and FMQuad pricing errors. From MPE, we notice that all models with IV in Group 1 as well as FBS model underestimate actual option prices and FMBAW and FMQuad overestimate actual option prices except OTM case. Considering HV, all models underestimate in call cases with the exception of FMBAW and FMQuad in the DITM case. Regardless of volatilities, all models perform best in the order of DITM, ITM, ATM, OTM, and DOTM when MAPE is considered, but the order is reversed if RMSE is considered. This result is consistent with the finding in Heo et al. (2010). One of the reasons is that the actual mean option values for DITM, ITM, ATM, OTM, and DOTM are $19.11, $6.92, $5.52, $4.61, and $2.45, respectively, so that even if RMSE of the DOTM options is smaller than that of the DITM options, the percentage error is much greater. We also notice that FBS yields almost the same MPE in the IV case, regardless of the moneyness. As we have seen before, all models in Group 2 outperform the corresponding models in Group 1. The improvement in the OTM and DOTM cases is significant. Usually models with IV outperform HV but models in Group 2 yield similar results in the DITM and ITM cases.

TABLE 4
MAPE WITH IV BY MATURITY AND MONEYNESS

			Group 1			Group 2		
Maturity	**Moneyness**	**N***	**B-S**	**MBAW**	**MQuad**	**FBS**	**FMBAW**	**FMQuad**
M_1	**DITM**	12400	2.5696	1.1697	1.1433	2.271	1.1142	1.121
	ITM	660	4.5891	3.4737	3.4094	2.8753	2.7755	2.8931
	ATM	860	5.7338	4.9127	4.9183	3.4234	3.4511	3.5373
	OTM	515	7.7734	6.9318	7.0299	4.16	4.2797	4.2825
	DOTM	852	9.6229	8.5281	8.8476	5.1317	5.0905	5.0781
	ALL	15287	3.4032	2.0839	2.0812	2.585	1.6456	1.6605
M_2	**DITM**	10980	6.4059	2.2237	2.0684	5.8207	2.0778	2.0214
	ITM	994	8.6586	5.6622	5.5273	6.9693	4.6226	4.6861
	ATM	1136	9.059	6.2727	6.3401	7.0786	5.0549	5.279
	OTM	870	9.6336	7.0935	7.3688	7.2112	5.6604	5.9905
	DOTM	4726	10.6471	8.4256	9.1278	7.9737	6.7209	7.1646
	ALL	18706	7.9084	4.4457	4.5417	6.5667	3.7335	3.8449
M_3	**DITM**	6165	8.8158	3.114	2.9076	7.9693	2.6569	2.3995
	ITM	747	10.8326	6.7323	6.7009	9.2506	5.3171	5.0555
	ATM	809	11.1031	7.3165	7.49	9.1776	5.7301	6.0146
	OTM	615	11.6594	8.1323	8.5214	9.8653	6.2024	6.6224
	DOTM	4952	12.5673	9.3872	10.4466	9.577	7.2162	8.935
	ALL	13288	10.5981	6.1434	6.4692	8.8018	4.8567	5.3999
M_4	**DITM**	10155	16.5799	5.4223	5.3374	1.4661	5.8355	4.9447
	ITM	1327	17.7113	8.7585	9.1718	1.7597	6.2784	5.2442
	ATM	1670	17.857	9.2948	9.9232	1.6125	6.5126	5.2992
	OTM	1239	17.9885	9.8829	10.7416	1.8914	6.3125	5.0389
	DOTM	13161	18.9403	11.3021	13.5456	3.0155	7.1186	5.0921
	ALL	27552	17.9027	8.8269	9.9639	2.2483	6.5322	5.0553

* **N** is the Number of Observations.

Table 4 presents MAPE estimation errors with IV across option maturities and moneyness. In Group 1, American pricing models, MBAW and MQUAD, perform better than their European counterparts across various moneyness and maturities. Surprisingly in Group 2, with maturity M_4, the European pricing model FBS's relative performance gets significantly better for all moneyness. Not only does FBS perform better than the other models, the magnitude of the error is much smaller despite the longer maturity. However, the results reported in Table 4 are not inconsistent with the results of Table 3. The performance of FBS for at ATM, OTM and DOTM cases in Table 3 due to its extraordinary performance in M_4.

CONCLUSION AND FUTURE STUDY

In this study, we examine the accuracy of two approximated solutions, denoted by FMBAW and FMQuad, to the fractional Black-Sholes partial differential equation by following the approaches of MBAW and MQuad, respectively. We empirically compare price errors using recent option data of petroleum industry - ConocoPhillips (COP), Chevron Corporation (CVX), Exxon Mobil (XOM) - traded on the Chicago Board of Trade Option Exchange (CBOE) from January 1, 2005 to December 31, 2009. Accuracy is measured with mean absolute percentage error with respect to option price (MAPE), mean percent error with respect to option price (MPE) and root mean squared error (RMSE) across different measures for moneyness, volatility, and option maturity. The American approximation models based on the fractional Brownian motion (fBm) are more accurate and reliable than the corresponding models, particular, with HV. In contrast to Ju and Zhong (1999), The MBAW performs better than MQuad. Overall, FMQuad seems to produces smaller error than FMBAW, but we find that when the maturity is less than 10 months (M_1, M_2, M_3) FMBAW is more accurate using MAPE. FMQuad is more reliable than the other models using RMSE regardless of volatilities. If the maturity is longer than 9 months (M_4), the FBS model noticeably outperforms the other models in all three measures.

This study is different from Heo et al. (2010), where the estimations were obtained by replacing B-S by FBS evaluating European option price, and to derive the critical stock price. However, both studies reveal some similar results with FBS model's overall better performance and as a good predictor of accuracy for American option pricing in this setting. As claimed in Heo et al. (2010), the FBS is a comprehensive choice for option pricing model for American option pricing as well as European option pricing.

The Hurst parameter is the key in studying fractional Brownian motion and our results also verify that the Hurst parameter contributes tremendously to pricing accuracy. Developing a more accurate method of calculating the Hurst parameter directly from the real data instead of recovering it from the FBS model as used in here would improve accuracy of prediction of option prices. It would be interesting to observe the pricing bias generated by different Hurst parameter estimation methods as described in Biagini, et al. (2008).

ENDNOTES

1. See Daye (2003) and Biagini, et al. (2008) for excellent references of fBM theory and applications.
2. Recently numerous studies compared and summarized various American option pricing models [Barone-Adesi, (2005); Pressacco et al., (2008); Li (2010)]. Notably Pressacco, et al. (2008) evaluated efficiency of option pricing techniques and obtained numerical results on American options with early exercise opportunity.
3. See equations (3) and (4) in Section MODELS for these moving boundary conditions.
4. According to two-tails mean difference t-test with a 95% confidence interval comparing the mean errors for the paired sample with corresponding models in Group 1 and Group2, P-values were less than 0.01 and we have sufficient evidence to claim that the models in Group 2 have less mean error than models in Group 1.
5. The numbers of observations for M_1(less than 3 months), M_2 (between 3 and 6 months), M_3(between 6 and 9 months), and M_4 (greater than 9 month) are 15287, 18706, 13288, and 27552, respectively.

6. According to error estimations conducted by option maturity of 30 day increment, this significant change occurs when the maturity is longer than 8 months in some cases.
7. The parameter values S=40, X=45, r=0.0488, σ=0.3, $\delta = 0$ and h=0.55 are used.
8. The numbers of observations for DITM, ITM, ATM, OTM, and DOTM are 39,700, 3,728, 4,475, 3,239, and 23,691, respectively

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APPENDIX

RMSE AND MPE WITH IV BY MATURITY AND MONEYNESS

			Group 1			Group 2		
Measure	**Maturity**	**Moneyness**	**B-S**	**MBAW**	**MQuad**	**FBS**	**FMBAW**	**FMQuad**
RMSE	M_1	**DITM**	0.5275	0.2752	0.2721	0.5031	0.2674	0.2663
		ITM	0.2457	0.19	0.1847	0.1543	0.1354	0.1391
		ATM	0.1925	0.1618	0.1611	0.1187	0.1131	0.1147
		OTM	0.1516	0.1346	0.1362	0.0878	0.086	0.0862
		DOTM	0.1028	0.0924	0.095	0.0628	0.0625	0.0623
	M_2	**DITM**	1.1426	0.4017	0.3824	1.0977	0.3882	0.3794
		ITM	0.4912	0.3408	0.3344	0.4185	0.2952	0.2988
		ATM	0.3957	0.2931	0.2952	0.3366	0.2559	0.2636
		OTM	0.3094	0.2419	0.2486	0.2571	0.2086	0.2196
		DOTM	0.1675	0.1386	0.1456	0.1393	0.1187	0.1247
	M_3	**DITM**	1.4633	0.5046	0.4741	1.3914	0.4493	0.4134
		ITM	0.7068	0.4684	0.4666	0.6228	0.3881	0.3795
		ATM	0.577	0.4062	0.4134	0.4912	0.3301	0.3497
		OTM	0.4821	0.3623	0.3747	0.4144	0.2889	0.315
		DOTM	0.2377	0.1872	0.1992	0.1973	0.1511	0.1891
	M_4	**DITM**	3.0711	0.984	0.9682	0.4543	1.089	0.9331
		ITM	1.7546	0.9102	0.9528	0.3368	0.6588	0.5645
		ATM	1.4985	0.8039	0.8564	0.2172	0.5671	0.4662
		OTM	1.3554	0.7764	0.837	0.2626	0.4999	0.4121
		DOTM	0.7207	0.4395	0.4943	0.1695	0.2802	0.2263
MPE	M_1	**DITM**	-2.5321	-0.9926	-0.9388	-2.1728	-0.7589	-0.7001
		ITM	-4.0173	-2.2933	-2.0284	-1.2234	0.3894	0.6528
		ATM	-4.3848	-2.8536	-2.793	-0.4822	1.0014	1.0805
		OTM	-6.0974	-4.7266	-4.89	-0.4371	0.9244	0.7857
		DOTM	-8.1251	-6.6466	-7.0868	-0.8809	0.5776	0.1361
	M_2	**DITM**	-6.4058	-2.1723	-2.0063	-5.7751	-1.7584	-1.6492
		ITM	-8.6543	-5.6197	-5.4805	-6.7673	-3.9878	-4.0028
		ATM	-9.0502	-6.2044	-6.2754	-6.7493	-4.1346	-4.2864
		OTM	-9.6112	-7.0016	-7.2889	-6.6711	-4.2523	-4.5269
		DOTM	-10.4076	-7.8998	-8.7378	-6.5739	-4.2622	-4.9692
	M_3	**DITM**	-8.8114	-3.084	-2.8731	-7.8981	-2.0375	-1.4035
		ITM	-10.8326	-6.7293	-6.6983	-9.0338	-4.161	-2.833
		ATM	-11.0649	-7.2658	-7.441	-8.9179	-4.2432	-2.9942
		OTM	-11.6594	-8.1225	-8.5139	-9.6996	-5.0524	-3.6557
		DOTM	-12.463	-9.1311	-10.2768	-8.8061	-3.7961	-3.406
	M_4	**DITM**	-16.5799	-5.4163	-5.3315	-0.2849	5.6157	4.6824
		ITM	-17.7113	-8.755	-9.1698	-0.2585	5.8582	4.7152
		ATM	-17.857	-9.2792	-9.9146	0.0979	6.3391	5.0435
		OTM	-17.9885	-9.8772	-10.7394	-0.2112	5.949	4.5378
		DOTM	-18.9032	-11.1382	-13.4608	-0.227	6.3887	3.8688