Risk Measurement-Value at Risk (VaR) Versus Conditional Value at Risk (CVaR): A Teaching Note

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Financial history has demonstrated that desirable and undesirable outcomes are always possible. Participants in the industry have made substantial progress in quantifying and mitigating many sources of risks. One more recent indicator is value at risk (VaR). However, this technique remains controversial, despite being an industry standard. Several studies have identified limitations with VaR. This teaching note compares it to conditional value at risk (CVaR) to demonstrate both the usefulness and limitations of these techniques and provide recommendations concerning which risk measurement method is more prudent under selected states of nature to aid professors in explaining the usefulness of VaR and CVaR.

INTRODUCTION

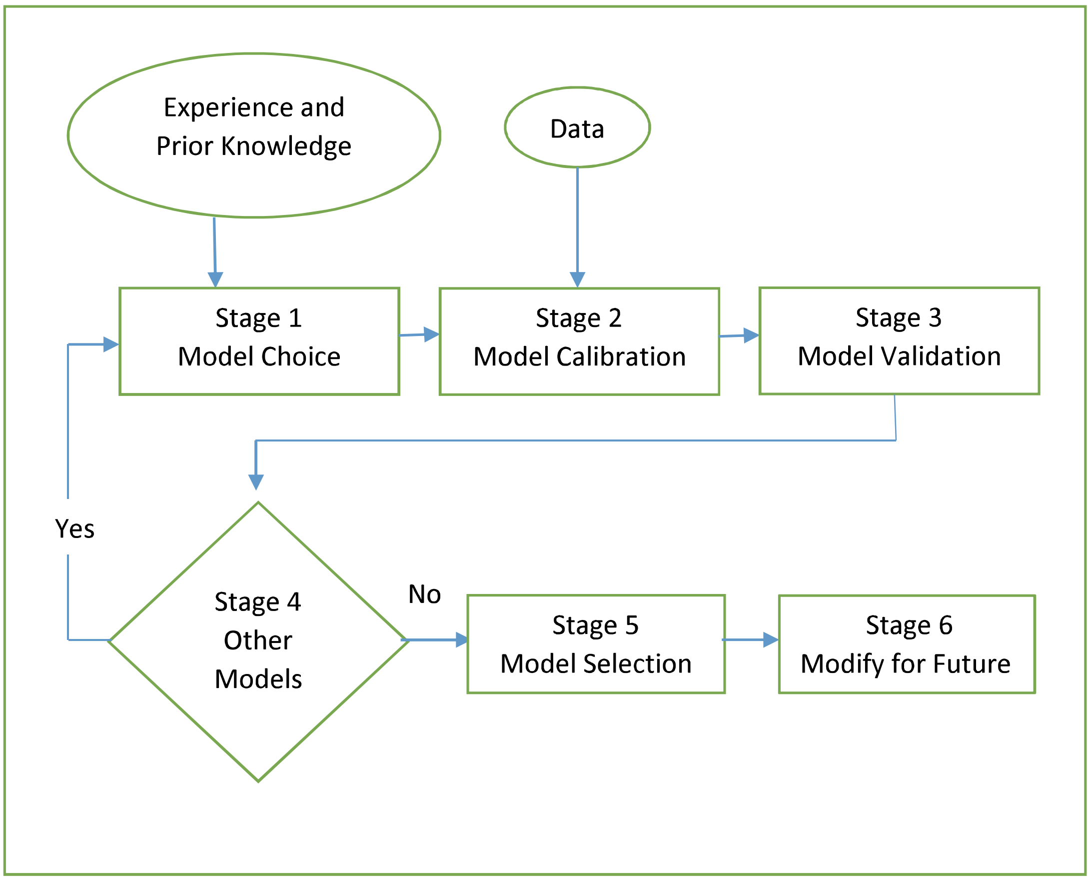
Measuring and managing risks are major concerns for financial institutions, practitioners and theoreticians. Financial risk is often defined as the degree of volatility inherent in a particular financial position. While the level of exposure to risk is usually measured by a number or a set of numbers, a more detailed and refined risk management plan is often desirable as it allows an institution or individual investors to undertake the most prudent approach to obtain its inherent financial position and mitigate effectively against the risk within that position. (Panjer, 1997)

Risk measurement typically involves a statistical approach to measure the likelihood of various events occurring, including the probabilities of the most likely and the most extreme events. From a risk management perspective, extreme events or the chances of adverse outcomes are much more interesting based purely on the historical (ex post) financial and operational consequences of such events.

Financial institutions generally consider three main categories of risks: market risk, credit risk and operational risk. Market risk encompasses the various risks of the market at a macro level, such as interest risk, foreign exchange risk, and volatility inherent in indices (VIX), as the some of the more salient risks. Credit risk is concerned with losses that arise due to default and bankruptcy. Lastly, operational risk deals with losses resulting from operational failures, such as an IT breakdown, failed internal control processes, and internal legal fraud. (Vaughan, 2002)

The development of probabilistic or stochastic models and their applications to measure the underlying risks have been of great importance to ensure that a fair and prudent determination of the required rate of return on capital can be estimated ex ante. Therefore, these models are of interest and importance to regulators, vendors of financial products and services, customers, and, of course, investors. Taking into account the potential domino effect within the financial industry, it is no surprise that there are experts who advocate the need to standardize parameters that are employed in the process of building probabilistic-based models. The process of modelling risk is a well-developed process that follows an accepted sequence (Panjer, 1997):

FIGURE 1
CONCEPTUAL PROCESS OF FINANCIAL RISK MODELLING

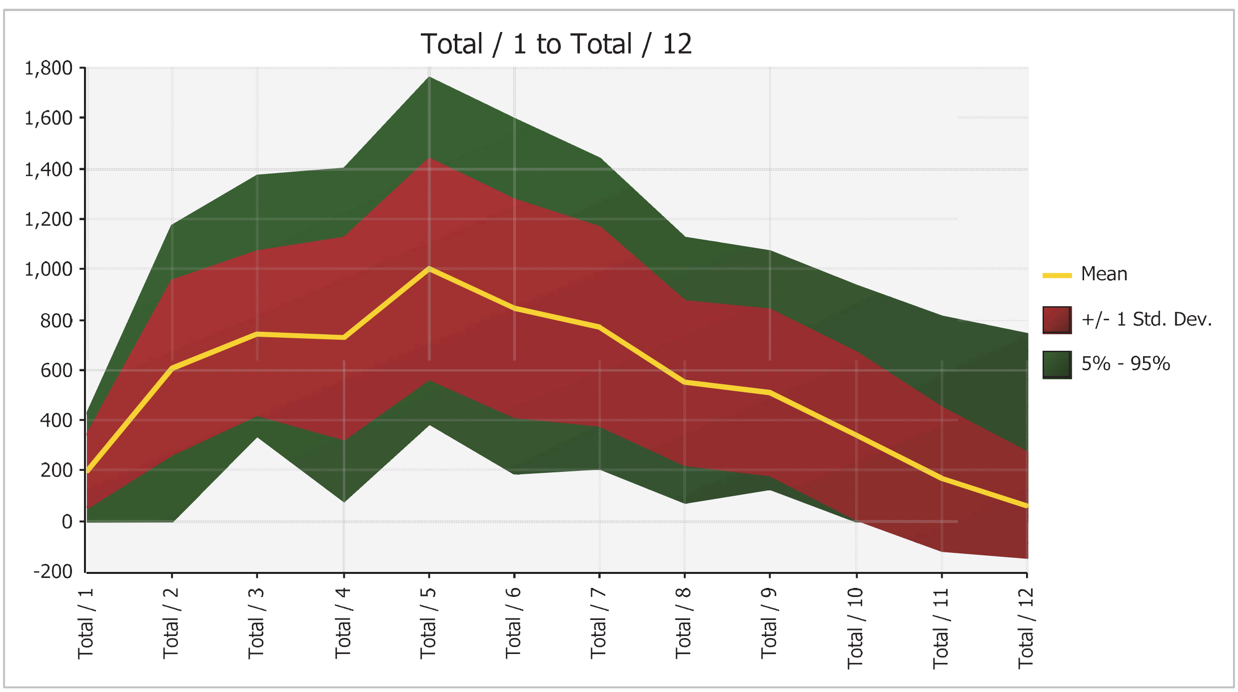


There are two major approaches to modelling and determining the capital required to maintain financial stability in the event of extreme outcomes. The first approach is to develop a mathematical model for each risk exposure separately and to assign a capital requirement to each exposure, based on the historical behavior of that particular exposure. This approach is normally referred to as the risk-based capital (RBC) approach. The total capital requirement is normally the sum of the capital requirement for each risk exposure, with a potential final adjustment to take in to account any interaction effects between exposures (correlation effects). Risks that are not perfectly correlated may have additional diversification or hedging advantages. Hence, the need for adjustment in the calculation of the total capital requirement.

The second approach takes a more rigorous view, which requires an integrated model for the entire organization. This approach is normally referred to as an internal model approach (IMA) with the requirement to develop a mathematical model to describe the entire organization, including all interactions among separate business units (SBU) and the various ancillary risks affecting the company, not unlike enterprise risk management (ERM). These internal relationships, including correlations, must be built into the model. This approach normally determines the total capital requirement for the organization as a whole, which is necessary to separately reallocate capital for each individual unit (Smithson, Smith, Jr, & Wilford, 1996).

An immediate question is the level at which the event is considered a risk or an extreme event. Under various financial scenarios, the answer may well vary. In this paper, the sources of the risks are numerous, ranging from market risk and credit risk to operational risks. So, it is inherently difficult to define a value that can adequately capture all types of risks in a single number or paradigm within a model. For the most part, however, an heuristic is that 90% of the distribution about the mean is the safe side. Hence, the 5% captured in each tail is considered to be a risky area and warrants closer scrutiny. In practice, the 5% tail on the right hand side is not usually considered risk, but more of an opportunity. Large loss gives way to large opportunity. A single sided confidence level of 95% includes the 5% tail on the right hand side and excludes the 5% tail on the left side. To demonstrate further, consider for example the cash flow distribution of the project below, with data provided courtesy of Palisade Corporation. Of the twelve years of project life, the upper and lower darker areas represent potential extreme payoffs. In this particular case, the extreme event might be a lower than expected cash flow (a risk) or higher than expected cash flow (an opportunity). That being said, both remain extreme events. The middle area represents 90% of the distribution around the mean and, therefore, is within expectation of being safe, assuming the standard convention of considering the 5% tail on the left hand side of the distribution to be indicative of extreme events requiring attention for risk mitigation.

FIGURE 2
A PROJECT CASH FLOW



PROBLEM STATEMENT

VaR provides a generic and convenient measurement of financial risk. Its unit of measurement is the same as the asset’s bottom line. It can, therefore, be easily communicated to a non-technically adept audience. Furthermore, VaR has achieved the status of being a commonly used metric in the regulation of various industries. Most notably, the Basel Accords have developed several frameworks and metrics to extensively regulate banking operations. These accords specifically employ the VaR measurement. VaR, however, remains a non-coherent and non-convex measurement and, as such, exhibits several limitations, especially when considering portfolio diversification where subadditivity of individual risks produces counter intuitive results. For instance, it is possible to construct a portfolio of two assets X and Y in a way that: $VaR(X+Y) > VaR(X) + VaR(Y)$. This particular portfolio, which can be expanded to consist of more than two assets, can produce results that are contradictory to the tenets of modern portfolio theory and lead to an incorrect portfolio diversification strategy. This particular scenario is generally regarded as a seminal example of the limitations in utilizing VaR to quantify risk.

In contrast to the increasing use of VaR, this paper further investigates the concept of a closely related variant, conditional value at risk (CVaR) and demonstrates that the use of CVaR may be more prudent and more appropriate under various states of nature for risk-averse investors. CVaR, in practice, does not receive the same level of recognition, despite its ability to provide mathematically superior results compared to VaR.

FINANCIAL RISK MEASUREMENT

VaR is a financial tool which has been widely used in the financial industry since its development in the late 1980s. VaR represents the amount of capital required by an enterprise to ensure that it will not become insolvent should an extreme event occur outside the pre-specified certainty. Consider a continuous distribution f(r) that represents the daily return of an asset/portfolio. The probability of the asset's return to be less than 5% on a particular day is:

$$F(0.05) = P(r \leq 0.05) = \int_{-\infty}^{0.05} f(r)dr \tag{1}$$

where $F(r)$ is a cumulative distributed function (cdf) which has the following properties:

$0 \leq F(r) \leq 1$ for all r
$f(r)$ is non decreasing

If, on the other hand, *f(r)* is discrete rather than continuous, the cdf takes the following form:

$$F(5) = P(r \leq 5) = \sum_{k}^{5} f(r=5) \tag{2}$$

where k is the lower limit of the discrete distribution.

In this sense, 5 or r_α is the return level where α% of the returns are greater than 5. From a mathematical point of view, VaR is the maximum level of loss corresponding to a certain probability of loss or to a given confidence interval within a given time frame (1 day in the previous example) (Hendricks, 1996). Therefore, given a value of the return level r_α, VaR can be described mathematically as (Kaura, 2008):

$$VaR_\alpha = sup\{r_\alpha \in \Re : F(r_a) \leq 1-\alpha\} = inf\{r_\alpha \in \Re : F(r_a) \geq \alpha\} \tag{3}$$

Despite being popular in the financial industry, VaR suffers from several other shortcomings.

If $f(r)$ is normally distributed, at a value of $k = 1.645$, the exceedance probability of $\Pr(X > \mu + k\sigma)$ = 5%. Similarly, if $k = 2.576$, then $\Pr(X > \mu + k\sigma)$ = 0.5%. Since the distribution is normally distributed, one would expect the probabilities of gains or losses to be equidistance from the mean, and have the same VaR. This situation remains true for general elliptical distributions for which the normal distribution is a special case. Once this underlying assumption of normally distributed no longer holds, at the same distance away from the mean, the exceedance probability would no longer be the same. In other words, in the case of non-elliptical distributions, one would expect to have two or more financial positions with the same VaR, but have different probabilities of failure.

Furthermore, VaR carries information on the tail boundaries of the distribution, but reveals little about the shape of the tails themselves. At the same VaR, a fat and heavy tail carries more risks than a long and slender tail. Hence, the cumulative losses or any other losses beyond the point of VaR are not reflected in VaR itself. It is, therefore, insufficient to refer to VaR as the sole measurement of risk. Some practitioners have a preference for CVaR. The mathematical definition of CVaR (Artikis & Artikis, 2012) is:

$$CVaR(\alpha) = \frac{\int_{-\infty}^{r_\alpha} r.f(r)dr}{\int_{-\infty}^{r_\alpha} f(r)dr} \tag{4}$$

CVaR answers a different question than VaR. CVaR indicates the average of the losses if the VaR point is exceeded or the average of the area under the curve is beyond the point of VaR. By calculating the area under the distribution curve from negative infinity to the VaR point, CVaR can measure and reflect irregularities inherent in the tail distribution. Based purely on its mathematical form, CVaR tackles loss functions that are continuous, whereas the VaR measurement is more static and discrete. In practice, loss functions more often display traits that belong to continuous distribution forms. Hence, CVaR is mathematically considered to be a more powerful measurement for losses (Artikis & Artikis, 2012).

In addition, VaR is also widely criticized on the grounds that it is not a coherent measurement of risk, which implies that accurate subadditivity is not possible. Systematically subadditivity is a very desirable property in financial management, as different investments pose different risks, and the risk of the overall portfolio can exceed the total sum of the individual risks under VaR measurement. The classification of coherent risk measurement must satisfy the following conditions (Panjer, 1997):

1. subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$
2. monotonicity: If $X \leq Y$ for all possible outcomes, then $\rho(X) + \rho(Y)$
3. positive homogeneity: for any positive constant c, $\rho(cX) = cp(X)$
4. translation invariance: for any positive constant c, $\rho(X + c) = \rho(X) + c$

VaR is well known to satisfy all but the subadditivity condition of the coherent risk conditions. Central to this non-coherent property of VaR is the fact that VaR itself is a quantile on the profit and loss function, but not the expectation. For that reason, the tail distribution before and after VaR have no bearing on its value. This property is inherently problematic for modelling financial risks, as the overall risk is not accurately measured due to the inability to use subadditivity and, hence, inaccurately hedged. Moreover, alternative investments using VaR may yield different and inaccurate risk measurements which, in turn, lead to inefficiency in the selection of different investments. Conversely, CVaR being an integral form of the function, satisfies all conditions of a coherent risk measurement. (Morini, 2011)

VaR AND CVaR PROCEDURES

The calculation of VaR and CVaR are highly similar. Consider a portfolio that consists of four investments with unequal weights and unequal returns on investments in Table 1:

TABLE 1
PORTFOLIO PARAMETERS

Portfolio weights (fractions of total investment)				
Investment	1	2	3	4
Weight	10%	25%	25%	40%
Parameters of (normally distributed) returns				
Investment	1	2	3	4
Mean	2%	5%	8%	14%
Std Dev	1%	3%	6%	12%

TABLE 2
PORTFOLIO CORRELATION MATRIX

Correlations Matrix	Return 1	Return 2	Return 3	Return 4
Return 1	1			
Return 2	0.7	1		
Return 3	0.7	0.7	1	
Return 4	0.7	0.7	0.7	1

The investment returns in consideration are simulated by normal distributions with the most likely returns (means) and their standard deviations in Table 1. The expected portfolio return based on weight is: E(r) = (10% × 2%) + (25% × 5%) + (25% × 8%) + (40% × 14%) = **9.05%**. The correlation matrix data provided in Table 2 are relatively stable and were obtained based on empirical evidence of historical data from a large selection of investments. However, the same correlation (0.7) was chosen in order to maintain the neutrality of the Monte Carlo simulation. Note that the calculation of VaR and CVaR can be verified through methods other than the stochastic Monte Carlo simulation, such as variance-covariance and historical simulation. In this paper, Monte Carlo simulation is used, following the process outlined by (Glasserman, Heidelberger, & Shahabuddin, 2008): generating a pre-specified N (10,000) scenarios by sampling change in the risk factors (returns) on a pre-specified time horizon (day); re-valuating the generated values at the end of the time horizon and subtracting the generated values from the current values (expected values); calculating and displaying the faction of each scenario both when the loss threshold is exceeded and not exceeded.

To perform a risk measurement of this portfolio, a Monte Carlo simulation of 10,000 iterations was conducted. The main intent of this simulation is to be able to conduct a large number of trials and derive the probability distribution for the portfolio returns. The worst 5% or 95% confidence in compliance with the standard practice is specifically highlighted in the simulated distributions. These probabilistic trials indicate a return value at 95% interval of -1.9% (Figure 3) implying that despite the positive expected returns of each individual portfolio, at 95% confidence, an investor of this portfolio is still subjected to a risk of obtaining a -1.9% return. As expected, the Monte Carlo simulation provides an expected portfolio return of 9.05%, which is the same as previously calculated. The simulation also indicates the maximum return being 33.616% and the minimum return (loss) is -15.226%.

CVaR, however, presents a different risk perspective for the investor. Based on the outlined methodology for computing CVaR, the 95% interval of the simulated distribution is filtered (Figure 4) to provide a closer examination of the remaining tail. Based on the theoretical definition of CVaR, the mean value of the tail in question is -4.59%. As this portfolio is comprised of multiple assets with varying more appropriate for a risk-averse investor. Financial regulatory bodies should also consider the use of CVaR given this more conservative measure of risk. As CVaR is more conservative, it should be used when the situation needs a conservative measure. It has reflected a higher loss at the same confidence interval than VaR. The situation is even more critical in the case of non-unimodal distributions, when at least one mode lies in the left tail. In such scenarios, CVaR would certainly generate more reliable results. Despite its mathematical superiority, CVaR has not yet received the same level of usage in the financial markets.

FIGURE 3
PORFOLIO VAR

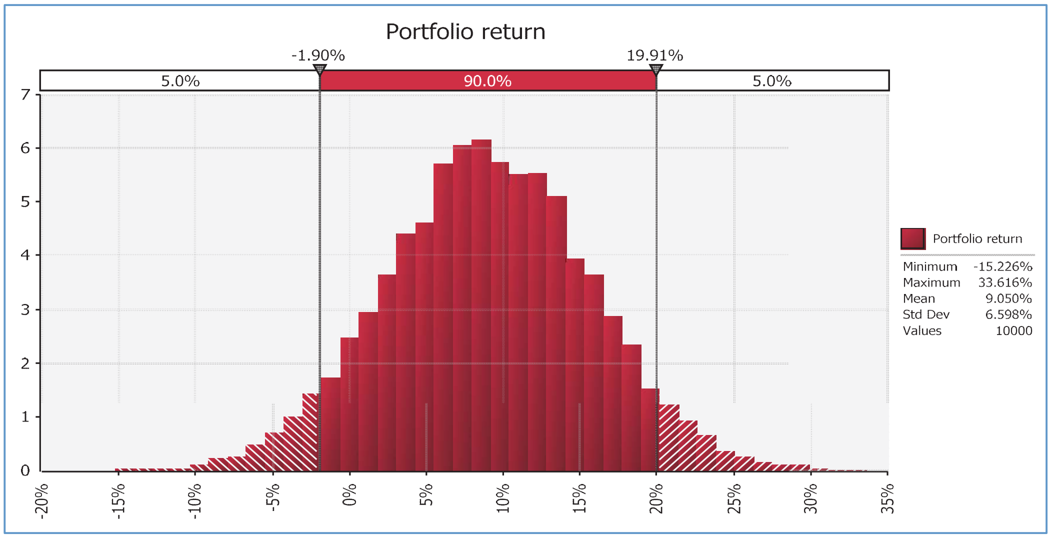
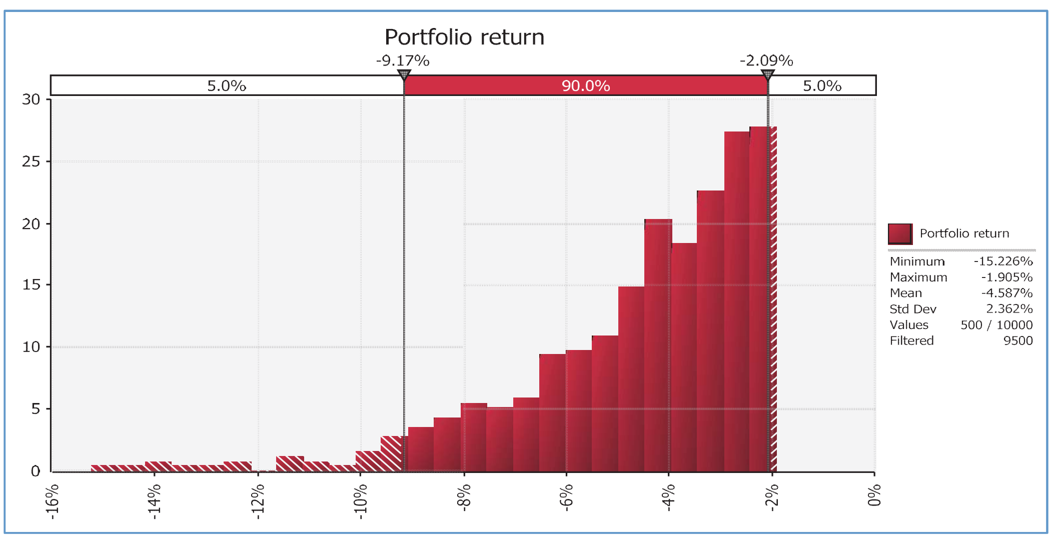


FIGURE 4
PORTFOLIO CVAR



CONCLUSIONS

While no single risk measurement and management strategy can neither provide nor predict a complete picture of the future potential losses, it is important to acknowledge that risk management provides a platform for future policy and strategic decision-making. The growing interest in financial risk management is not driven solely on the historical losses, but also because risk management is a management technique that helps investors and financial service providers gain insights into the behavior of many important financial instruments.

On a practical note, VaR undoubtedly has been an industry standard to measure risk. However, mathematically, CVaR certainly has some clear advantages and, therefore, should not be neglected. Ultimately, the investor's degree of risk aversion should determine the preference for either using VaR or CVaR in practice. More importantly, VaR is not an entirely coherent measure of risk as CVaR cannot accurately compute risk when assets are added to a portfolio. As demonstrated through the portfolio risk analysis in this paper, CVaR is a more conservative measurement, especially in the case of multiple assets. Even in the case of a single asset, CVaR still has the ability to more accurately depict loss scenarios in the case of non-elliptical distributions. Considering the characteristics of both CVaR and VaR, it has been shown that CVaR has a stronger potential to help an investor in making more informed investing decisions than Var.

ACKNOWLEDGEMENTS

The authors thank the participants at the 2018 MBAA-International Academy of Finance conference for their valuable comments on an earlier version of this paper, which received a *Best in Track Award* at this meeting. Special thanks go to Thomas Krueger and Heather Knewtson for their invaluable comments in improving this manuscript. Memorial University of Newfoundland's grant accounts (Hossain no. 211811 and 209392) provided financial support for completion of this project.

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