An Operational Goal Programming Model of Mutual Fund Portfolio Determination

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An operational investment model is designed and built in order to find the optimal combination of a set of pre-screened mutual fund investments to make over a specific time horizon, subject to the goals and constraints defined by the user. The model can recommend an optimal investment mix based upon the user's goals with respect to desired return, desired risk exposure, Sharpe Ratio, or a combination thereof.

INTRODUCTION

The purpose of this model is to find the optimal combination of initial and recurring investments into a given set of candidate mutual funds over five years, subject to certain goals and constraints. Sixteen 'no-load' funds were pre-selected by the user from two different fund families, Royce and USAA. The candidate funds were chosen for consideration based upon historical performance data. A model with many more funds and/or a distinct set of candidate funds may be appropriate to another investor; customization to the situation of the individual investor is straightforward, while retaining the basic design tenants used here.

This model uses goal programming to determine the allocation of assets among the 16 different mutual funds over five years based upon three goals: maximizing return, minimizing risk, and maximizing the Sharpe Ratio for the portfolio. These three goals are weighted in the objective function, using user-defined coefficient values to establish their relative importance. The model's constraints formally account for return, risk, Sharpe Ratio, budget, minimum investments and diversification. The budget for the entire five year period is set at $200,000, with the initial investment pool of $100,000 on the first day of year one, followed by annual investment pools of $25,000 on the first days of years two through five.

The portfolio also must have a diversity of funds. These investor-specific requirements state that no more than 70% of the budget can be invested in Royce funds, no more than 60% of the budget can be invested in USAA funds, at least 20% of the budget must be invested in small cap funds, and at least 12% of the budget must be invested in international funds. The model also requires that at least eight different funds are invested in during each period.

The model accounts for minimum investments by ensuring that the investments meet the criteria for each individual fund's minimum entry requirement; it then ensures that subsequent investments into that fund meet the fund's minimum additional allocation requirement. The model is submitted to the Simplex method, as implemented in the optimization software LINDO.

LITERATURE REVIEW

Optimization of investment portfolios based upon multiple criteria, such as risk and return, has been a topic of research for over fifty years. Since his early work framing portfolio selection as a mean-variance optimization problem with an efficient frontier, Harry Markowitz is generally regarded as the founding father of Modern Portfolio Theory (MPT). Prior to his seminal work, there was no formal attempt to quantify portfolio risk. Markowitz (1952) proposed a method by which an investor could assemble a portfolio which would generate returns greater than the mere weighted summation of the returns of the individual components. Markowitz contended that diversification of a portfolio has a direct affect on the risk level of the portfolio (Aouni, 2009). His mean-variance framework argued that, when building a portfolio, the co-movement of assets with each other is often more important than the individual security attributes. He introduced measurement of the dependence structure of returns, via the variance-covariance matrix, and quantified the impact on portfolio selection of each asset vis-à-vis the others; thus, formalizing the arguments for diversification and the 'efficient portfolio'.

Understanding and implementing the underlying statistical implications of the Markowitz optimization model presents challenges. Several approaches have been put forth to address the problems of estimating the many elements in the covariance matrix [Ledoit and Wolf, (2004); Jagannathan and Ma (2003); Chan *et al* (1999); Chopra (1993); Green and Hollifield (1992)]. DeMiguel *et al* (2009) develop a framework for establishing an optimal portfolio in the presence of estimation error for the means and covariances of asset returns. They solve the traditional minimum variance problem, based on sample covariances, but subject to an additional constraint that the norm of the portfolio-weight vector be smaller than a stated threshold. Sharpe (1972) proposed a single-index model that circumvented the issues with implementing the quadratic formulation of the Markowitz efficient frontier. He suggested a linear framework that provides the foundation for a linear programming approach to portfolio optimization.

In formal optimization models, portfolio diversification is typically imposed via specific constraints limiting how much can be allocated (in actual dollars and/or percentages) into certain types of investments (Abdelaziz, Aouni, & Fayedh, 2007). In other models, binary variables are employed to impose quantity constraints (Mamanis & Anagnostopoulos, 2010). Diversification requirements such as these will be incorporated into the current model. Binary variables will play a role in setting the minimum number of chosen funds, as well as helping to establish the minimum investment allowable for each fund.

Markowitz's bi-criteria portfolio selection model has evolved into multi-criteria models that include more than two objectives (Abdelaziz, Aouni, & Fayedh, 2007, Aouni, 2009; Mamanis & Anagnostopoulos, 2010). This research has brought about the creation of stochastic dominance models, multi-attribute utility models, multi-objective programming models, discriminant analysis, heuristic models, neural networks, optimization models, and multi-criteria decision aid methods (Aouni, 2009). Recently, the realization that adding more objectives to the model enhances its usefulness and dependability has motivated researchers to incorporate new criteria into selection models. These multi-dimensional models have helped to curb the assumptions that must be made by bi-criteria models, such as Markowitz's (Mamanis & Anagnostopoulos, 2010).

Many portfolio models have failed to incorporate the preferences of the investor (Aouni, 2009). Investor preferences are to be explicitly included in the model, in order to account for the difference in acceptable risk and return (Davies, Kat, & Lu, 2005; Abdelaziz, Aouni, & Fayedh, 2007). This has shown to have significant effects on the reliability of the models. One model, built around hedge funds, seeks to optimize multiple conflicting objectives: maximizing expected return, minimizing return variance, maximizing skewness and minimizing kurtosis. Davies, Kat, & Lu (2005) show that the preferences of the investor have a profound impact upon the allocation of assets across the selection of hedge funds. Because of this, investor preference will be heavily incorporated into this optimization model for mutual funds via the coefficients in the objective function along with the constraints specific to the individual.

Further, the multi-criteria model of this paper is designed with multiple periods of investment. Some models are created to give recommendations over a specific time span, with investments made periodically throughout (Yu, Takahashi, Inoue, & Wang, 2010). This will be incorporated into this model via a time span of 5 years, with investments being made at the beginning of each year, following an initial investment in selected funds.

Sharpe (1966) developed a metric (then called 'reward-to-variability') for evaluating and predicting the performance of mutual fund managers. His capital asset pricing model (CAPM) extended the Markowitz theory to an equilibrium theory. The Sharpe Ratio measures performance relative to volatility, which is essentially the goal of the entire bi-criteria model. This value expresses the performance of a fund in relation to its risk (volatility) and can be used in a multi-criteria model to further establish a solid, dependable portfolio. The Sharpe Ratio will be the third parameter in the objective function for this model, establishing an even greater balance between risk and return. After nearly 50 years, this 'Sharpe Ratio' remains one of the most popular indexes of performance. The portfolio with the maximum Sharpe Ratio is the portfolio with the highest reward-to-risk ratio achievable from the available assets.

Nevertheless, alternatives to the Sharpe Ratio have periodically been advocated for portfolio selection [Choueifaty and Coignard, 2006; Szego, 2004; Pedersen and Satchell, 2002; Ortobelli, et al, 2003; Farinelli and Tibiletti, 2003; Uryasev, 2000; Sortino, 2000]. Unfortunately, these theoretically valid metrics seldom recommend the same portfolio solution, leaving the practicing investor at a loss as how to proceed. The Sharpe ratio, and it variations, have been criticized when the assumption of normality in the distribution of returns is relaxed; It can result in incorrect investment decisions when the returns exhibit kurtosis or skewness [Leland 1999]. Farinelli *et al* 2008 propose various asymmetric parameter-dependent ratios and a robustness test to affirm the merits of selected alternative ratios vis-à-vis the Sharpe ratio. This study will assume the normal distribution of asset returns and, thus the appropriateness of the Sharpe ratio. Moreover, following Martellini (2008), this study returns the focus to the true optimal weighting scheme consistent with modern portfolio theory, i.e. pursuit of the highest Sharpe Ratio.

ASSUMPTIONS

Due to issues of manageability and forecasting, there are certain assumptions that must be made explicit.

1. <u>Investments</u>: The entire amount of money available to be invested during any year is allocated into the portfolio. There is no option to "not invest" any funds as the entire amount of money available to be invested during any year will be invested at the very beginning of the year.
2. <u>Returns</u>: The rates of return in this model are assumed to be normally distributed, and based upon past performance (historical five–year average returns) tempered by user-developed projections of future performance. For the purposes of this study, the future return estimates are partially determined by expected changes to the federal funds rate.
3. <u>Withdraws</u>: After money is invested into a fund, it cannot be removed during the balance of the five year period.
4. <u>Risk Value</u>: The risk value is calculated as a beta risk weighted average value.
5. <u>Sharpe Ratio</u>: The Sharpe Ratio is calculated as a weighted average Sharpe Ratio value.
6. <u>Future Risk</u>: The current beta risk for any fund is the same for that fund through all five periods.
7. <u>Future Sharpe Ratio</u>: The current Sharpe Ratio for any fund is the same for that fund through all periods.
8. <u>Transaction Costs</u>: There are no transaction costs of any kind for these no-load funds.
9. <u>Scale</u>: Due to scaling issues with the optimization software, the unit of value for money is scaled down to 1:10,000. This means that a value of 1 is equal to $10,000.

VARIABLE DEFINITIONS

Control Variables

The continuous-valued control variables all represent a monetary contribution to a specific fund within a specific fund family during a specific time period. Each variable name is made up of three parts: the fund family, the specific fund, and the time period, e.g. "**ROY**" "**1**" "**A**". The first part, "**ROY**", corresponds to the fund family "Royce". (The other option is "**USA**", which would correspond to the "USAA" fund family.) The second part, "**1**", corresponds to a specific fund within that fund family (Royce or USAA). Each family has eight funds participating in this model; therefore the numbers **1**, **2**, **3**, **4**, **5**, **6**, **7**, and **8** will be used. The third part, "**A**", represents the time period for that fund's investment. This model is set over a five year period, with one investment each year. "**A**" corresponds to investment/year 1, and "**B**", through "**E**" are used to reflect investment years two through five respectively. Thus, the variable "**ROY1A**" represents the monetary contribution to Royce family, fund 1, during year 1. The set of 80 control variables is: **ROYij** and **USAij** where i = 1 to 8 and j=A to E.

Switch Variables

Switch Variables are restricted to the binary values, 0 and 1. These variables are used as a "switch" mechanism to show that a certain requirement either has, or has not, been met. This model uses two different sets of switch variables (sets **S** and **V**). The switch variables are modeled exactly the same as the control variables, except they have the additional notation on the end in order to identify them as a switch variable, e.g. **ROY1A_S** or **ROY1B_V**.

"S" Switch Variables

The "S - set" of switch variables is the most important set of switches in this model. This set of switch variables is designed to show whether a fund during a given period has been allocated any money or not. This means that if the control variable for a certain fund during a given period has a value greater than zero, its corresponding "S" switch variable must be equal to "1", otherwise, it must be equal to "0". The "S" switch variables are used to implement portfolio diversification, minimum allocation amounts, as well as determining the values of the other set of switch variables. Thus, the full set of 80 companion "S" switch variables is **ROYij_S** and **USAij_S** where i = 1 to 8 and j=A to E.

"V" Switch Variables

The "V" switch variables have one purpose in this model; that is to set the minimum amount allowed to be invested in any fund during a given period. Many funds have a higher minimum allocation for the first investment into that fund than for all subsequent investments. The "V" switch variables help to write constraints that satisfy these requirements. The "V" switch variables require the value of their corresponding control variables to fall within a certain acceptable range, i.e. either zero *or* some amount at or above the minimum amount for that fund. As opposed to the control variable set and the "S" switch variable set, the "V" switch variable set does not include any variable for period "A". This is because investment values for period "A" will always be either 0 or *at least* the start-up amount. There is no need to use a "V" switch variable to distinguish between the start-up minimum allocation and the additional investment minimum allocation during period "A". Thus, the full set of 64 companion "V" switch variables is **ROYij_V** and **USAij_V** where i = 1 to 8 and j=B to E.

Goal Programming 'Deviation' Variables

The investor begins by identifying his investment objectives and translating these into specific target values for each of return, risk, and the Sharpe Ratio. The intent is to minimize the weighted sum of the following three undesirable deviations.

PR_UND: (Percent Return Underachievement): Continuous Variable

This variable captures the underachievement of the Percent Return target. The intent is to seek greater return; therefore minimizing the Percent Return Underachievement will help accomplish this goal. Thus: **PR_UND =** % Return Goal – Realized % Return
R_OVR: (Risk Overachievement): Continuous Variable
This variable captures the overachievement of the Risk target. The intent is to reduce total risk exposure; therefore minimizing the Risk Overachievement will help accomplish this goal. Thus: **R_OVR** = Realized Total Risk - Risk Goal
SR_UND: (Sharpe Ratio Underachievement) : Continuous Variable
This variable captures the underachievement of the Sharpe Ratio target. The intent is to increase the portfolio's Sharpe Ratio; thus minimizing this deviation variable will help accomplish this goal. Thus: **SR_UND =** Sharpe Ratio Goal – Realized Sharp Ratio

Other Variables and Parameters

Goal Programming necessitates the specification of 'stretch-goal' targets and accompanying deviation variables. The undesirable deviations were acknowledged above; the desirable deviations and user-specified target parameters and definitional variables that facilitate design are acknowledged below:

TR: (Total Return) : Continuous Variable
PR_OVR: (Percent Return Overachievement) : Continuous Variable
PR_GOAL: (Percent Return Goal) : Continuous Parameter
TRISK: (Total Risk) : Continuous Variable
R_UND: (Risk Underachievement) : Continuous Variable
R_GOAL: (Risk Goal) : Continuous Parameter
TSR: (Total Sharpe Ratio) : Continuous Variable
SR_OVR: (Sharpe Ratio Overachievement) : Continuous Variable
SR_GOAL: (Sharpe Ratio Goal) : Continuous Parameter
BUDGET: (Budget) : Continuous Parameter

In sum, the complete set of variables includes continuous dollar denominated control/decision variables (n=80), binary switch variables (S: n=80; V: n=64), deviation variables (n=6), for a total of 230 variables and four definitional parameters (TR, TRISK, TSR, BUDGET).

THE CONSTRAINING ENVIRONMENT

Constraints restrict either individual variables or groups of variables to a range of values that is applicable to the environment and/or user. Goal Programming includes two types of constraints: traditional, or 'hard', constraints and goal, or 'soft', constraints. Traditional constraints utilize a "=", "<" or ">" relational operator, without deviation variables, in order to ensure that the value of that constraint satisfies a stated condition without compromise. In contrast, goal/soft constraints use only the "=" operator, and include a pair of deviation variables to allow for possible deviation from the user-specified right-hand side target value. For example:

2X + 4Y + Z_UND – Z_OVR = Z

Control Variables (2X, 4Y) Deviation Variables (Z_UND, Z_OVR) Target Value (Z)

This 'soft' constraint has added the pair of deviation variables "**Z_UND**" and "**Z_OVR**", allowing the constraint to deviate, either above or below the target value. This is useful when the objective function solution could get greater benefit from a value which is just above or below the targeted value.

The goal value for a soft constraint must be either a number value or a variable that has a specific number value assigned to it elsewhere in the model. Goal programming relies upon soft constraints to provide goals and deviation variables to measure the realized discrepancies from such targets. Such deviations provide the focus of the optimization.

The purpose of the model is to find the optimal value for the objective function while satisfying each hard constraint and coming as close as possible to each target value of the soft constraint(s). The constraints used in this model are divided into three different groups: i) Goal Constraints, ii) Switch Variable Constraints, and iii) Control Variable Constraints.

Goal Constraints

These are three sets of constraints that operate together and directly affect the value of the objective function. These constraints establish the stretch-goals for the model and introduce one pair of deviation variables. In each set, there is one traditional L.P. constraint, one soft constraint, and one parameter-setting definitional statement.

Return Function Set

This constraint set provides the model with the necessary latitude to seek the highest possible percentage return on investment. The first constraint is a traditional constraint that pairs each control variable with its expected percentage return at the end of the 5 year period. For example, "**1.3136522 ROY1A**" means that the money allocated to the fund "**ROY1**" during period "**A**" is multiplied by "**1.3136522**" in order to find its expected value at the end of the 5 year period. The coefficient of "**1.3136522**" means that by the end of the 5 year period, the value of that variable is expected to grow by **31.36522%**. This constraint then adds together the expected values of all the control variables to find the portfolio's total return value at the end of the 5 year period; the variable "**TR**" holds that value.

The next constraint in this set is the soft constraint. It takes the portfolio's total return value (**TR**), as calculated by the previous constraint, and multiplies it by ".05" to find the percentage return on the entire portfolio. This constraint then introduces the deviation variables "**PR_UND**" and "**PR_OVR**" from the percent return goal, the value of which is set in the constraint following this one. Thus, this constraint is:
.05 TR + PR_UND - PR_OVR - PR_GOAL = 0

The third equation merely sets the goal for total portfolio percentage return at the end of the five year period. The variable was created in the previous constraint, but its value is set here as: **PR_GOAL = 1.5**

Risk Function Set

This set enables the model to pursue the lowest possible portfolio average risk. The first constraint pairs each control variable with its expected beta risk value through the end of the 5 year period. For example, "**1.11 ROY1A**" means that the money allocated to the fund "**ROY1**" during period "**A**" is multiplied by its beta risk of "**1.11**". This is done to find that variable's risk value. This constraint then adds together the risk values of all the control variables to find the portfolio's total risk value, and creates the variable "**TRISK**" to hold that value.

The next constraint in this set is the associated soft constraint. It takes the portfolio's total risk value (**TRISK**), as calculated by the previous constraint, and multiplies it by ".05" to find the weighted average beta risk of the entire portfolio. This constraint then introduces the deviation variables "**R_UND**" and "**R_OVR**" for the portfolio's risk goal, "**R_GOAL**": **.05 TRISK + R_UND - R_OVR - R_GOAL = 0**
The third constraint sets the goal for the weighted average beta risk of the entire portfolio. The parameter was created in the previous constraint, but its value is set here: **R_GOAL = .6**

Sharpe Ratio Function Set

This constraint set enables the model to pursue the highest possible portfolio Sharpe Ratio. The first constraint pairs each control variable with its expected Sharpe Ratio through the end of the five year period. For example, "**.28 ROY1A**" means that the money allocated to the fund "**ROY1**" during period "**A**" is multiplied by its Sharpe Ratio of "**.28**". This is done to find that variable's Sharpe Ratio value. This constraint then adds together the Sharpe Ratio values of all the control variables to find the portfolio's total Sharpe Ratio value, and creates the variable "**TSR**" to hold that value.

The next constraint in this set is the accompanying soft constraint. It takes the portfolio's total Sharpe Ratio value (**TSR**), as calculated by the previous constraint, and multiplies it by ".05" to find the weighted average Sharpe Ratio of the entire portfolio. This constraint then introduces the deviation variables "**SR_UND**" and "**SR_OVR**" which represent the deviation, under or over, from the portfolio's Sharpe Ratio goal, "**SR_GOAL**": **.05 TSR + SR_UND - SR_OVR - SR_GOAL = 0**

The next statement sets the goal for the weighted average Sharpe Ratio of the entire portfolio. The parameter was created in the previous constraint, but its value is set here as: **SR_GOAL = .75**

Switch Variable Constraints

These constraints provide the structure for determining the value of each binary switch variable.

Constraints for Switch Variable "S"

Two sets of constraints are needed to determine the value of each "S" switch variable. One set fixes the value to "1" and the other set of constraints sets the value to "0". These mutually exclusive constraints are designed so that a switch variable can never equal both "1" and "0" simultaneously.

"S" Switch "ON" Constraints

These constraints require the "S" switch variable to take on a value of "1" whenever its corresponding control variable has a positive value. The maximum value for any control variable is less than 11, so for any feasible value of a control variable, subtracting 11 from it would make it less than zero, satisfying the constraint. The general form of these 80 constraints appears as:

$$\mathbf{XXX_{i,j} - 11XXX_{i,j}\_S < 0}$$

where XXX is either ROY or USA, and i=1 to 8, and j= A to E.

"S" Switch "OFF" Constraints

These constraints force the "S" switch variable to be "0" when its corresponding control variable is "0". If the control variable is equal to "0", subtracting .001 from it would make it less than zero, not satisfying the constraint. The general form of these 80 constraints appears as:

$$\mathbf{XXX_{i,j} - .001XXX_{i,j}\_S > 0}$$

where XXX is either ROY or USA, and i=1 to 8, and j= A to E.

Constraints for Switch Variable "V"

The set of "V" switch variables bases their values on the values of the "S" switch variables. If the fund corresponding to the "V" switch variable has been allocated money in past periods, then the value of the "V" switch variable must be "1".

"V" Switch "ON" Constraints

These constraints require the value of a "V" switch variable to be "1" if the value of any "S" switch variable from the same fund during <u>any</u> previous period is equal to "1". Representative of the full set of 64 such constraints are the four constraints relevant to ROY1x below:

ROY1A_S - 2 ROY1B_V < 0
ROY1A_S + ROY1B_S - 3 ROY1C_V < 0
ROY1A_S + ROY1B_S + ROY1C_S - 4 ROY1D_V < 0
ROY1A_S + ROY1B_S + ROY1C_S + ROY1D_S - 5 ROY1E_V < 0

"V" Switch "OFF" Constraints

The "OFF" constraints worked by requiring the "V" switch variable to be "0" so long as the "S" switch variables from the corresponding fund for all previous periods were equal to "0" also. Representative of the set of such constraints are the four constraints relevant to ROY1x below:

ROY1B_V - ROY1A_S < 0
ROY1C_V - ROY1B_S - ROY1A_S < 0
ROY1D_V - ROY1C_S - ROY1B_S - ROY1A_S < 0
ROY1E_V - ROY1D_S - ROY1C_S - ROY1B_S - ROY1A_S < 0

Control Variable Constraints

These constraints provide structure for interaction among the control variables. The constraints may include switch variables; but in order to be included in this category, the constraint must have a direct impact upon the acceptable values of the control variables.

Constraints Limiting Budget

This set is made up of multiple constraints, one to define the budget for each period.

Total Budget

This definitional constraint sets the amount of money that will be invested throughout all periods. It creates the parameter "**BUDGET**" to hold that value as: **BUDGET = 20**

Period Budgets

These constraints set the amount of money that will be invested across the available funds during the specified period. Period "A" has a budget of "10" ($10,000), while all other periods have a budget of "2.5" ($2,500).

Constraints for Portfolio Diversification

These constraints ensure that the portfolio meets certain diversification requirements of the specific investor. This means that the constraints set either a ceiling or floor value on a certain group of funds so that each falls within an acceptable percentage window of the total budget.

Royce Funds: This limits the value of all investments into the Royce fund family to 70% of the budget.
USAA Funds: This limits the value of all investments into the USAA fund family to 60% of the budget.
Small Cap Funds: This sets the collective floor of investments into small cap funds to 20% of the budget.
International Funds: This sets the collective floor of investments into international funds to 12% of the budget.

Minimum Number of Funds Allocated Money Each Period

This group of five constraints uses the set of "S" switch variables to require that at least eight different funds are invested in during each period. (The number 'eight' is user-specified.)

Constraints for Minimum Investment Amounts

These constraints represent a minimum investment to a fund during its designated period. The constraints require the variable to be either equal to zero, or greater than a certain value. These constraints relate to initial investment minimums and recurring periodic investment minimums.

Constraints for Minimum Initial Investments Amounts

These constraints account for the "start-up" investment into a fund. Most funds have a higher minimum initial investment than they do a minimum additional investment. Using the sets of "S" and "V" switch variables, these constraints require the value of a variable to be at least the minimum initial investment into its designated fund, or zero. For example: **ROY1B + .2 ROY1B_V - .2 ROY1B_S > 0**

This constraint applies to the fund **ROY1** during period "**B**". The minimum initial investment to this fund is .2 units ($2,000). If there is no investment into the fund (**ROY1B** = 0), then **ROY1B_S** must inherently equal zero, thereby immediately satisfying the constraint regardless of the value of **ROY1B_V**. Only when the value of the control variable (**ROY1B**) is greater than zero does the "V" switch variable play any role. Assuming **ROY1B** is greater than zero (forcing **ROY1B_S** = 1), the **ROY1B_V** variable can either allow the .2 unit minimum investment to be enforced (**ROY1B_V** = 0), or counter it and have no minimum investment (**ROY1B_V** = 1). The value of the "V" switch variable is determined by the value of **ROY1B** in all previous periods. Representative of this collection of 80 constraints is the following for ROY1x:

ROY1A - .2 ROY1A_S >0
ROY1B + .2 ROY1B_V - .2 ROY1B_S > 0
ROY1C + .2 ROY1C_V - .2 ROY1C_S > 0
ROY1D + .2 ROY1D_V - .2 ROY1D_S >0
ROY1E + .2 ROY1E_V - .2 ROY1E_S > 0

Constraints for Minimum Additional Investment Amounts

These constraints account for the additional investment into a fund. Most funds have a lower minimum additional investment than they do a minimum initial investment. Using the set of "S" switch variables, these constraints require the value of a variable to be at least the minimum additional investment into its designated fund, or zero. Representative of this collection of 64 constraints is that for ROY1B: **ROY1B - .005 ROY1B_S > 0**

This constraint applies to the fund **ROY1** during period "**B**". The minimum additional investment is .005 units ($50). If there is no investment into the fund (**ROY1B** = 0), then **ROY1B_S** must default to zero.

There are no constraints pertaining to period "A" in this set as this is the first opportunity to invest in any fund, so the minimum initial investment amount applies to every fund in period A.

In sum, this model is built with 428 total constraints of which 425 are the traditional 'hard' constraints of linear programming, and three are the 'soft' type, incorporating deviation variables, unique to goal programming.

OBJECTIVE FUNCTION

The goal of this model is to minimize the deviation from specified values of three different goals. The model simultaneously seeks to maximize return by minimizing the underachievement of the return goal, minimize risk by minimizing the overachievement of the risk goal, and maximize the portfolio's Sharpe Ratio by minimizing the underachievement of the Sharpe Ratio goal. There are three variables in the objective function that relate to these goals of the model. These are PR_UND (percent return underachievement), R_OVR (risk overachievement), and SR_UND (Sharpe Ratio underachievement). These variables are assigned the coefficients of 1.8, 1.2, and 1 respectively, that reflect the user-defined relative importance of each goal in the model. The formal objective function is:

MIN 1.8PR_UND + 1.2R_OVR + SR_UND

CONCEPTUAL MODEL

The conceptual model seeks to maximize the percentage return after 5 years, minimize the weighted average beta risk of the portfolio, and maximize the weighted average Sharpe Ratio of the portfolio, subject to constraints that account for return, risk, Sharpe Ratio, budget, diversification, and minimum investments. Specifically, the conceptual model (including investor-specific parameters) appears as:

Minimize (underachievement of ending percentage return goal + overachievement of weighted average beta risk goal + underachievement of weighted average Sharpe Ratio goal)

Subject to:

$\sum fund_{i,j} * return_{i,j} + d_1^- - d_1^+$ = total return goal (where d_1^- and d_1^+ are deviation variables for return)

$\sum fund_{i,j} * risk_{i,j} + d_2^- - d_2^+$ = total risk goal (where d_2^- and d_2^+ are deviation variables for risk)

$\sum fund_{i,j} * Sharpe\ Ratio_{i,j} + d_3^- - d_3^+$ = total Sharpe Ratio goal (where d_3^- and d_3^+ are deviation variables for Sharpe)

Total Budget = 200,000

Year 1 investment = 100,000

Subsequent investment caps of $25,000 annually in years 2 through 5

$\sum$Royce $fund_{i,j} \leq .7$ * Budget

$\sum$USAA $fund_{i,j} \leq .6$ * Budget

$\sum$Small Cap $funds_{i,j} \geq .2$ * Budget

$\sum$International $funds_{i,j} \geq .12$ * Budget

Number of funds allocated money each year ≥ 8

Investment in $fund_{i,j} \geq$ minimum investment for $fund_{i,j}$ (minimum investment for $fund_i$ changes after initial investment is made into $fund_i$)

OPTIMAL SOLUTION and INTERPRETATION

The optimization results (after after 10,113 iterations) are summarized in Table 1 below. The recommended solution gives the dollar amount to be allocated into each fund during each period. The objective of the model is to minimize the value of the collective weighted deviations, thereby obtaining the maximum return with the minimum risk and the maximum Sharpe Ratio.

The minimal objective function value of .8917752 doesn't have meaning beyond that of a measure of collective deviation and for comparison with other possible solutions. No user targets were completely met. The table above shows what the actual value of each deviation variable is in comparison to what the goal for the variable was. The goal for percent return was 150% and the actual return was 128%, therefore giving an underachievement value of 22%. For risk, the goal was a .6 beta point weighted average and the actual risk was .87725, making the overachievement value .27725. The Sharpe Ratio goal was .75 and the actual Sharpe Ratio was .61289, giving it an underachievement value of .13711.

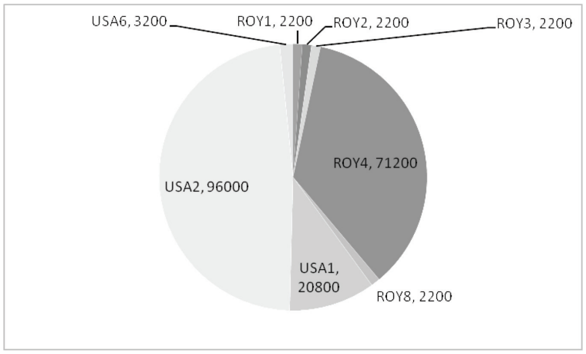
TABLE 1
OPTIMAL INVESTMENT ALLOCATION

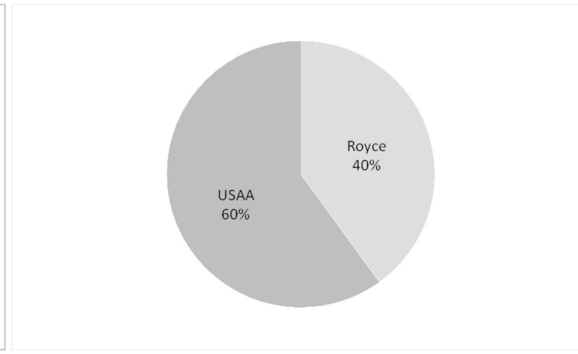
	Year 1	Year 2	Year 3	Year 4	Year 5
ROY1	$ 2,000.00	$ 50.00	$ 50.00	$ 50.00	$ 50.00
ROY2	$ 2,000.00	$ 50.00	$ 50.00	$ 50.00	$ 50.00
ROY3	$ 2,000.00	$ 50.00	$ 50.00	$ 50.00	$ 50.00
ROY4	$ 70,800.00	$ 100.00	$ 100.00	$ 100.00	$ 100.00
ROY5	$ -	$ -	$ -	$ -	$ -
ROY6	$ -	$ -	$ -	$ -	$ -
ROY7	$ -	$ -	$ -	$ -	$ -
ROY8	$ 2,000.00	$ 50.00	$ 50.00	$ 50.00	$ 50.00
USA1	$ 3,000.00	$ 17,650.00	$ 50.00	$ 50.00	$ 50.00
USA2	$ 15,200.00	$ 7,000.00	$ 24,600.00	$ 24,600.00	$ 24,600.00
USA3	$ -	$ -	$ -	$ -	$ -
USA4	$ -	$ -	$ -	$ -	$ -
USA5	$ -	$ -	$ -	$ -	$ -
USA6	$ 3,000.00	$ 50.00	$ 50.00	$ 50.00	$ 50.00
USA7	$ -	$ -	$ -	$ -	$ -
USA8	$ -	$ -	$ -	$ -	$ -

	Percent Return	128%
PR_GOAL	Percent Return Goal	150%
PR_UND	Percent Return Goal Underachievement	22%
PR_OVR	Percent Return Goal Overachievement	0%
	Risk	0.87725
R_GOAL	Risk Goal	0.6
R_OVR	Risk Goal Underachievement	0.27725
R_UND	Risk Goal Overachievement	0
	Sharpe Ratio	0.61289
SR_GOAL	Sharpe Ratio Goal	0.75
SR_UND	Sharpe Ratio Goal Underachievement	0.13711
SR_OVR	Sharpe Ratio Goal Overachievement	0

Figure 1 shows that eight of the 16 possible funds will be allocated money, five from the Royce family and three from the USAA family; it also depicts how much of the total budget is being allotted to each fund or fund family.

FIGURE 1
DISTRIBUTION AMONG FUNDS

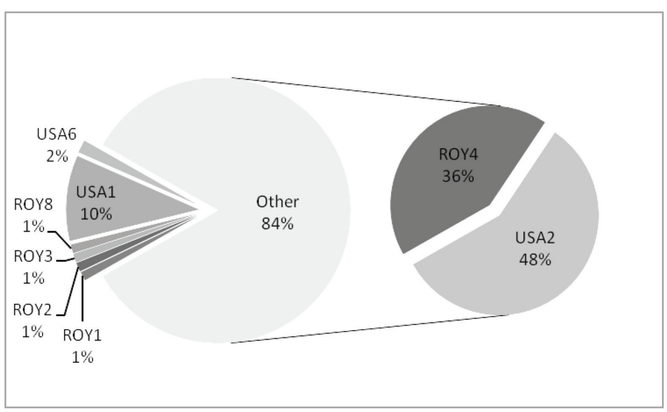




USAA funds received 60% of the total budget, equal to $120,000. This was the limit for the fund family. Within this family, three funds were allocated money, with one fund, USA2, receiving most of it. This fund received almost half of the entire budget. The Royce funds received 40% of the budget, equal to $80,000. This was spread out over five funds, with one fund, ROY4, receiving most of it. The other four funds all received the minimum amount necessary to be deposited into those funds.

It seems that the portfolio is almost entirely made up of two funds, with a few others added in just to meet the mandates of the model. These two funds account for 84% of the portfolio value as depicted in Figure 2 below. This would be a poor investment decision for someone who seeks the safety of diversity among funds. If these funds perform poorly during any given period, it would have catastrophic effect of the entire portfolio. Conversely, if these two funds perform abnormally well during any given period, it would greatly increase the value of the portfolio. This combination on funds has increased the risk of the portfolio in a way that is not measured by the model. Typically this risk is accounted for in the constraints, preventing this type of investment diversification.

FIGURE 2
CONCENTRATION WITHIN THE PORTFOLIO



The small cap funds were allocated $75,600. The international funds requirement was met exactly, with a total $24,000 being invested into two international funds, USA1 and USA6. $200,000 was invested in the portfolio, and at the end of the five years the value of the portfolio is $256,670.53. Risk is less than a weighted average beta value of 1, which means the portfolio is less volatile than the market average; the Sharpe Ratio is quite favorable .61289.

SENSITIVITY ANALYSIS

In the presence of any form of integer variables, the routine automated sensitivity analysis cannot be completed. Because of this, the "revise and resubmit" method was used to perform a sensitivity analysis of selected parameters. The purpose of this analysis is to determine the extent of the volatility of the results when certain objective function coefficients or right hand side values are changed.

Range of Optimality of Objective Function Coefficients

In order to determine the range of optimality, the coefficient of a deviation variable in the objective function was changed, and then the model was solved and analyzed to see if the change had impacted the basis. The point at which the changed coefficient affects the basis is the end of the range of optimality. This was done repeatedly until a lower and upper bound coefficient was established for the coefficient of each deviation variable, as summarized in Table 2 below.

TABLE 2
RANGE of OPTIMALITY of OBJECTIVE FUNCTION COEFFICIENTS

Range of Optimality		
Variable in Objective Function	Lower Bound Coefficient	Upper Bound Coefficient
Percent Return	0.85	1.83
Risk	1.09	1.52
Sharpe Ratio	0.76	1.04

The ranges of optimality for this model show that the model has a large range for a decrease in the coefficient of percent return (.95), but has very small range for increasing it (.03). The same can be said for the Sharpe Ratio coefficient; the range for increase is also very small (.04), while the range for decrease is much larger (.24). The risk coefficient, on the other hand, has a larger range for increase (.32) than it does for decrease (.11).

The impact of different coefficients upon the value of the three driving variables is established by solving the model with three different coefficients for one variable, all other things kept the same. The three coefficients were determined by adding "1" and subtracting "1" from the original coefficient. Since the coefficient cannot be zero, the Sharpe Ratio low coefficient is set to ".01". The results of these changes are shown in Figures 3, 4 and 5 below.

When the PR_UND coefficient was modified, Figure 3 reveals that the values of the three driving variables were all impacted. The low coefficient caused a 2% drop in return, while the high coefficient made little difference in the return, only increasing by .5%. The impact on risk was similar, with its value dropping by .03 with the low coefficient and rising only by .01 with the high coefficient. The Sharpe Ratio was caused to drop in both situations, although by only a third as much with the high coefficient than the low coefficient.

FIGURE 3
IMPACT of CHANGING PERCENT RETURN COEFFICIENT

PR Coefficient	Return	Risk	Sharpe Ratio
.8	1.26346835	0.84469	0.59265
1.8	1.28335265	0.87725	0.61289
2.8	1.28821525	0.885702	0.606935

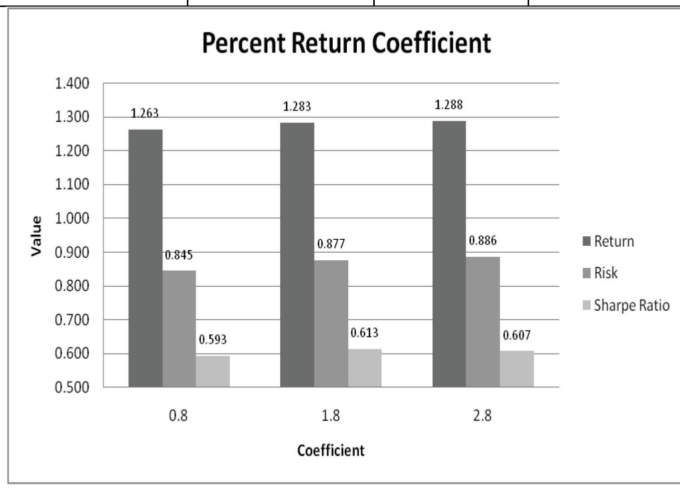
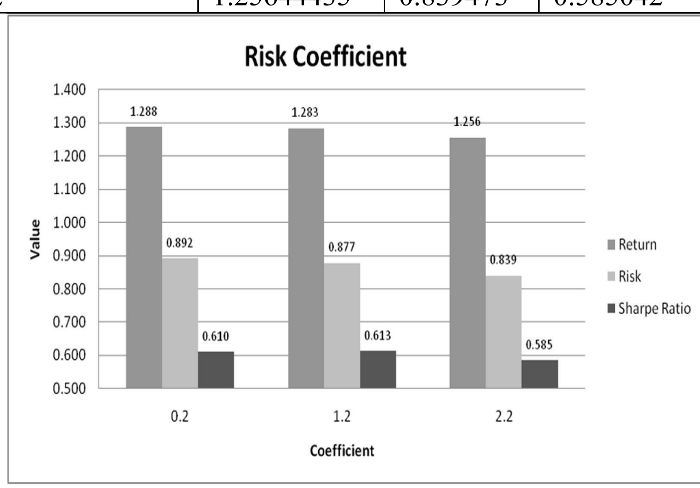


FIGURE 4
IMPACT of CHANGING RISK COEFFICIENT

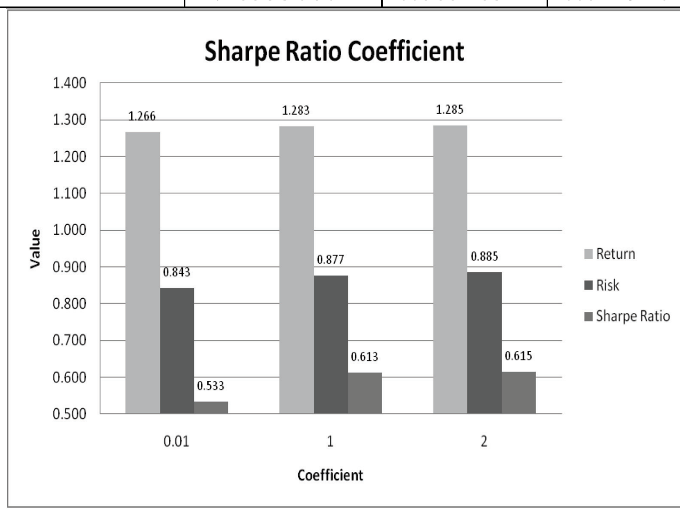
R Coefficient	Return	Risk	Sharpe Ratio
0.2	1.28753655	0.891885	0.609887
1.2	1.28335265	0.877250	0.612890
2.2	1.25644435	0.839473	0.585042



As Figure 4 above summarizes, when the R_OVR coefficient was modified, the values of the three driving variables were again all impacted. The low coefficient made little difference in return, causing a .4% increase, while the high coefficient decreased the return by almost 3%. The impact on risk was similar, with its value increasing by .014 with the low coefficient and decreasing by .04 with the high coefficient. The Sharpe Ratio was again caused to drop in both situations, yet this time the large coefficient resulted in the larger change. The value decreased by almost .03 with the larger coefficient, while the lower coefficient only decreased it by .003.

FIGURE 5
IMPACT of CHANGING SHARPE RATIO COEFFICIENT

SR Coefficient	Return	Risk	Sharpe Ratio
0.01	1.26611260	0.842553	0.533342
1	1.28335265	0.877250	0.612890
2	1.28533680	0.885183	0.614940



When the SR_OVR coefficient was modified, the values of the three driving variables were once again all impacted. The low coefficient made larger difference in return, causing almost a 2% decrease, than the high coefficient, which increased the return by only .2%. The impact on risk was similar, with its value decreasing by .035 with the low coefficient and only increasing by .008 with the high coefficient. The impact on the Sharpe Ratio was much different than in previous changes. The low coefficient caused the value to drop by almost .09, three times its previous largest change. Furthermore, with the larger coefficient, the Sharpe Ratio increased for the first time, even though the increase was small at .002.

Summary of Coefficient Range Analysis

Table 3 below summarizes the above impact analysis. These results suggest that Return has an overall resistance to change when coefficient values are altered. There also does not seem to be high volatility to the value of Risk when coefficients are changed. With the exception of one value, the Sharpe Ratio values all seemed somewhat stable as well. Obviously the largest changes in value came when the coefficient of the respective variable was the one being changed. The variables were much more volatile in decreases than increases. This is mostly likely due to the method of selecting new coefficient values. Had the coefficient values been selected in a way that reflects a similar percentage change, rather than adding or subtracting one, the increases and decreases are likely to be very similar to each other. Those results would show even less volatility in these variables than these results do, indicating that the model is relatively stable regardless of coefficient values.

TABLE 3
SUMMARY OF COEFFICIENT CHANGE ANALYSIS

Value	Return	% Change	Value	Risk	% Change	Value	Sharpe Ratio	% Change
Highest	1.28753655	0.33%	Highest	0.891885	1.67%	Highest	0.614940	0.33%
Original	1.28335265	0%	Original	0.877250	0%	Original	0.612890	0%
Lowest	1.25644435	-2.10%	Lowest	0.839473	-4.31%	Lowest	0.533342	-12.98%

Range of Feasibility / Right Hand Side Range Analysis

The right hand side value for the maximum percent of the portfolio that can be invested into Royce funds was varied from its original RHS value. This RHS value was changed to .5, .6, .8, and .9 and compared to the original output at .7. The results are shown in Table 4 below.

This analysis reveals a few things about the volatility of the model during changes to this specific RHS value. The original RHS value is .7, so the model was re-solved using two values below that value and two values above it. When the model was solved with the RHS value at .5, return increased, risk decreased, and the Sharpe Ratio increased. With a RHS value of .6, there was no change from the values found at .5. This means that this change from .5 to .6 did not go outside of the range of feasibility for that RHS value. The change from .6 to .7 did affect the solution however, meaning that .7 is not within the range of feasibility for the model. At .7, return decreased, risk increased, and the Sharpe Ratio decreased. When solved at .8, the solution changed once again. Return increased, risk increased, and the Sharpe Ratio decreased. When solved with an RHS value of .9, the solution changed yet again. At .9, return decreased, risk increased, and the Sharpe Ratio increased.

TABLE 4
ANALYSIS of VARYING a SELECTED RHS PARAMETER

Max Investment	Return	Risk	Sharpe Ratio
0.5	1.28335265	0.87725	0.61289
0.6	1.28335265	0.87725	0.61289
0.7 (baseline)	1.28282815	0.87953	0.61259
0.8	1.28492575	0.87962	0.60619
0.9	1.28142635	0.88033	0.611518

The results show correlation between rising maximum allocation into a fund family and rising risk. This suggests that the risk levels of the funds being invested in from the Royce family are greater than those of the funds in the USAA family. Also, there seemed to be a negative correlation between return and the Sharpe Ratio. This is because the Sharpe Ratio incorporates risk into its value, and the typical relation of risk and return is indirect. Regardless of the correlation, the changes that actually occurred due to the changes in RHS ranges were small and had little effect on the value of the objective function.

CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

This study was intended to create a model that could recommend an investment portfolio seeking to provide an investor with the return, risk, and Sharpe Ratio they seek. It has shown great promise in being able to do so. The model is deficient in certain aspects such as fund selection and diversification requirements. With a larger model, these things can be more efficiently accounted for and this model could be of great use to an investor.

Investing will never be an exact science, as this model attempts to make it, because of the unknown future returns, risk levels, Sharpe Ratio's, and other measures of performance. However, with well informed projections and estimates of future values, this model can be invaluable to the average investor. The model accounts for the investor's desire for returns and risk aversion. A person without experience in financial markets who seeks to invest would have a much better chance of making wise investments using this model because it can account for issues that the investor cannot on his own.

There are some deficiencies in the model that were discovered during the process of analyzing the data. First, the constraints within the model do not efficiently force diversification of the portfolio. The results from this model show heavy investment into only a few funds. This tells the investor that the funds are superior to the others based upon the investor's preferences, however it adds risk to the portfolio indirectly in a way that is not measured by the model. The diversification requirements would be decided by the investor, however it is suggested that more diversification be applied than is in this model.

Perhaps the most important continuance is to expand the model to include a greater selection of funds. Sixteen funds is a very small segment of the financial market; expanding the model would provide the investor with more options for investment. Another possibility for this model is to add a way to customize the time periods for investments, whether by increasing the length of time or having multiple investment periods per year. This would allow an investor to gain an even better perspective on the portfolio that they are creating. This would also be beneficial for those who are looking to put money into an investment for a long period of time.

It would be very beneficial to this model to add the ability to remove money from one fund and place it into another. As returns and risks change, investors may want to reinvest money previously allocated into one fund, and reallocate it to another, more advantageous fund. This will add invaluable flexibility to this model, and almost turn the model into a personal financial advisor. This could become an operational tool for investors, assuming that the model can receive updated data pertaining to returns, risk, etc. on a regular basis.

REFERENCES

Abdelaziz, F. B., Aouni, B., & Fayedh, R. E. (2007). Multi-Objective Stochastic Programming for Portfolio Selection. *European Journal of Operational Research,* 177, 1811-1823.

Aouni, B. (2009). Multi-Attribute Portfolio Selection: New Perspectives. *INFOR,* 47-1, 1-4.

Chan, L., Karceski, J., & Lakonishok, J. (1999). On Portfolio Optimization: Forecasting Covariances and Choosing the Risk Model. *Review of Financial Studies,* 12, 937-974.

Chopra, V. (1993). Improving Optimization. *Journal of Investing,* 8, 51-59.

Davies, R. J., Kat, H. M., & Lu, S. (2005). *Fund of Hedge Funds Portfolio Selection: A Multi-Objective Approach.*

DeMiguel, V., Garlappi, L., Nogales, F., & Uppal, R. (2009). A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms. *Management Science,* 55-5, 798-812.

Farinelli, S., Ferreira, M., Rossello, D., Thoeny, M., & Tibiletti, L. (2008). Beyond Sharpe Ratio: Optimal Asses Allocation Using Different Performance Ratios. *Journal of Banking and Finance,* 32, 2057-2063.

Farinelli, S. & Tibiletti, L. (2003). Upside and Downside Risk with a Benchmark. *Atlantic Economic Journal,* 31-4, 387.

Green, R. & Hollifield, B. (1992). When Will Mean-Variance Efficient Portfolios Be Well Diversified? *Journal of Finance,* 47, 1785-1809.

Grover, J. and Lavin, A. (2007). Modern Portfolio Optimization: A Practical Approach Using an Excel Solver Single-Index Model. *The Journal of Wealth Management,* 10-1, 60-73.

Jagannathan, R. & Ma, T. (2003). Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *Journal of Finance,* 58, 1651-1684.

Leland, H. (1999). Beyond Mean-Variance: Performance Measurement in a Non-Symmetric World. *Financial Analysts Journal,* 55, 27-35.

Mamanis, G., & Anagnostopoulos, K. (2010). A Portfolio Optimization Model with Three Objectives and Discrete Variables. *Computers and Operations Research*, 37, 1285-1297.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance,* 7-1, 77-91.

Martellini, L. (2008). Toward the Design of Better Equity Benchmarks: Rehabilitating the Tangency Portfolio from Modern Portfolio Theory. *Journal of Portfolio Management,* 34-4, 34-41.

Ortobelli, S. Huber, I., Rachev, S. & Schwartz, E.(2003). Portfolio Choice Theory and Non-Gaussian Distributed Returns. In S.T. Rachev, ed. *Handbook of Heavy Tailed Distributions in Finance,* Amsterdam, Elsevier, 547-594.

Pedersen, C. and Satchell, S. (2002) On the Foundation of Performance Measures under Asymmetric Returns. *Technical Report*, ComBridge University.

Sharpe, W. (1966). Mutual Fund Performance. *Journal of Business,* January, 119-138.

Sortino, F. (2000) Upside-Potential Ratios Vary by Investment Style. *Pensions and Investments,* 28, 30-35.

Szego, G. (2004). *Risk Measures for the 21st Century.* Chichester, Wiley & Sons.

Uryasev, S. (2000). Probabilistic Constrained Optimization Methodology and Applications. Dordrecht; Kluwer Academic Publishers.

Yu, M., Takahashi, S., Inoue, H., & Wang, S. (2010). Dynamic Portfolio Optimization with Risk Control for Absolute Deviation Model. *European Journal of Operational Research*, 201; 349-364.