The Useful Algebraic Formulae for Interest Computation in Financial Decisions

Keishiro Matsumoto
University of the Virgin Islands

Robert W. Hull
Western Illinois University

Christy Vineyard
University of the Virgin Islands

Sharon A. Simmons
University of the Virgin Islands

A new theory of term loans is proposed and proved in this paper. The algebraic formulae to compute the loan balances of popular term loans and their interest expenses are derived from the theory. Excel users can utilize the formulae in calculating tax savings on accrued interest in refinancing analysis without undue hardship. Furthermore, it is typically necessary to conduct a sensitivity or simulation analysis since key variables involved in financial decisions are often subject to uncertainty. The new formulae will make the use of an amortization schedule obsolete in computing interest expenses. The new method should be of great value in pedagogic settings, since computational tedium is replaced by a more intellectually stimulating effort to understand the logic behind the formulae. It is also indispensable for software engineers developing computer programs to solve application problems in these areas because the formula approach absolutely supersedes in numerical efficiency the traditional method based on an amortization schedule.

I. INTRODUCTION

The concept of generalized term loans (henceforth G-term loans for brevity) will be introduced in this work. Since the terminology is a new one, it is necessary to provide an expository discussion here. A generalized term loan is a term loan whose payments need not remain fixed during the term of a loan. The balance of a G-term loan might rise or decline prior to its maturity but it has to reach zero at maturity.[1]

The G-term loan comprises all popular term loans utilized in business today, such as an ordinary term loan, a deferred term loan, an annuity due, a balloon loan, a bullet loan, and an interest-only loan.[2] For exposition of popular term loans, see introductory finance texts such as Keown et al. (2010). The reason for grouping them into a new category is because a theory established for the group will hold true for any of its members.

This work is concerned with annual interest expenses when underlying loans are any one of the G-term loans. The reason is that the assessment of tax savings on annual interest expenses is a critical factor, for instance, especially in mortgage refinancing analysis. The traditional method to compute the latter is to generate monthly interest expenses by using an amortization schedule and sum them up year by year to obtain the latter. Tax savings are computed by multiplying annual interest expenses by a firm's tax rate. The traditional method of computing annual interest expenses by means of an amortization schedule is particularly laborious even aided by a spreadsheet program. The alternative method to be presented in this work develops the algebraic formulae to calculate interest expenses for all G-term loans. The algebraic formulae will be of great value in making financial decisions in bond refunding and lease or purchase, mortgage refinancing, and capital budgeting analysis. For examples, see respectively Keown et al. (2010), Chen (1997), and Harris and Pringle (1985).

A major advance in financial education in the last decade is the use of Excel in teaching introductory finance. Jackson and Staunton (2001) are early writers who recognized the value of Excel in financial education. However, its rapid acceptance in financial education clearly reflects the success of Principles of Finance with Excel by Benninga (2005) from Oxford University Press.[3]This work was followed by his Financial Modeling (2008) from MIT Press. Excel appears to have been well received by researchers in financial education and text writers. See Whitworth (2010). Further, researchers report that students whose instructors use Excel in introductory finance perform better than those whose instructors do not use Excel. See Cagle, Glasgo, and Hyland (2010). MacDougall and Follows (2006) conclude that students trained with Excel are more likely to be better managers because their understanding of subject matters is deeper. It appears that Excel is viewed as an effective tool of finance pedagogy.

The emergence of Excel in financial education is a highly timely development for the algebraic formula approach to interest computation. Since finance students today are competent Excel users, they can readily design an Excel worksheet to conduct refinancing analysis, for instance, by storing algebraic formulae on cells of
their worksheet.

However, the use of an amortization schedule to compute accrued interest, especially in refinancing analysis, is not a viable option for even average Excel users. The reason is that transferring interest expenses from an amortization schedule to a refinancing worksheet is not a simple task. Consequently, the computational effort to obtain the net advantage of refinancing is bound to become mentally and physically overwhelming when the term of a mortgage loan is long.[4]

In this regard, it should be pointed out that a feature of refinancing analysis also shared by other capital budgeting problems is the need to follow up an initial analysis by a sensitivity analysis, since variables such as future interest and tax rates are subject to uncertainty. It is often necessary to investigate the range of the rates for which one decision is preferred to another by conducting sensitivity analysis or to investigate the shape of the relative frequency of accepting refinancing by conducting a simulation experiment when the new interest rate is by assumption stochastic, for instance. See Matsumoto et al. (2011) for an example of an Excel refinancing analysis as well as an Excel simulation analysis under the algebraic formula approach to interest computation. The traditional method based on an amortization schedule is ill-suited for the latter due to the lack of numerical efficiency. Whereas, with the algebraic formula approach, it is possible to conduct even a simulation experiment on an Excel worksheet.

With this background in mind, the purpose of the paper is to present a general theory to derive a loan balance formula for any term loan and the formula to compute annual interest accrued so that tax savings on the latter can be efficiently calculated. The paper is an analytical work that both finance students and finance professionals will find highly useful in enhancing computational efficiency. Although it neither addresses any deep philosophy in financial economics nor reports a novel empirical discovery in financial behavior, this work is a true contribution to refinancing analysis.

The organization of the remainder of the paper is as follows. Section II formulates the rigorous definition of the G-term loans and sets forth the key property of G-term loans. Section III describes their amortization schedules of G-term loans and asserts its existence. Section IV presents the main theorem of this work. Section V relates how the loan balance formulae are derived. Section VI depicts exactly how the annual interest formulae are obtained. The final section VII is for the summary and concluding remarks.

II. THE KEY DEFINITION AND PROPERTY

Let the monthly interest rate on a mortgage loan be i x 100% and n its term. Let B_t, P_t, C_t, A_t, and B_{t+1}, respectively denote its t-th beginning balance, the t-th payment, the t-th interest cost, the t-th amortization, and its ending balance of the period t.[5] However, it should be kept in mind that the latter is also the t+1-th beginning balance of this loan. With the new notations introduced, the next step is to state the definition.

A. The G-Term Loan

It is necessary to specify the condition which makes a term loan a G-term loan in this work. The following is the mathematical condition required of a G-term loan.

Definition 1:
The G-loan is a term loan for which the following condition holds:

$$B_1 = \frac{P_1}{(1+i)^1} + \frac{P_2}{(1+i)^2} + \ldots.. + \frac{P_t}{(1+i)^t} + \ldots + \frac{P_n}{(1+i)^n} \tag{1}$$

and the loan balance must reach 0 only at maturity♦[6]

Let us state the key property of G-term loans of interest next.

B. The Key Property

The key property here is a necessary condition of a G-term loan. That is, if there is no amortization schedule, a loan cannot be called a G-term loan.

Necessary Condition:
A G-term loan must have its own amortization schedule♦

III. AMORTIZATION SCHEDULES

The amortization schedule of an ordinary term loan is always discussed in a time value of money chapter in an introductory finance text. However, no discussion of other G-term loans is available in a standard finance text. It is necessary to describe the amortization schedule of the G-term loans for clarity. The reason is that the new theory of this work is built on the premise that all G-term loans have their own amortization schedule. In the remainder of this section, all amortization schedules will be presented. Additional comments will be provided whenever needed, especially for a G-term loan which is relatively unknown. Let us next describe that of the G-term loan per se.

A. An n period i × 100% G-term loan next.

The amortization schedule of a G-term loan consists of the six columns. They are a period subscript t, t-th loan balance, t-th payment, t-th interest cost, t-th amortization, and t-th ending balance which is also the beginning balance of the next period. The top row contains the label of each column. Therefore, the total number of rows is set to n+1 for all G-term loans except a deferred term loan. The total number of rows is n+ d+1 for the latter.

Consider an arbitrary period whose period subscript is denoted by t. Its t-th payment P_t is not required to be a level constant for a G-term loan. Its t-th interest cost is the product of the t-th loan balance and the interest rate: $C_t = iB_t$. Its t-th amortization is obtained by subtracting the interest cost from the payment: $A_t = P_t - C_t$. The t+1-th balance B_{t+1} is derived by subtracting the amortization from the beginning balance: $B_{t+1} = B_t - A_t$. The entry in the last column of row t is B_{t+1}.

Let us consider the last row n. Since the loan balance B_{n+1} on the last row must reach 0, the payment P_n must be the sum of the loan balance B_n and the interest cost $C_n = iB_n$. See Table 1 below for the amortization schedule.

TABLE 1
AN AMORTIZATION SCHEDULE OF A G-TERM LOAN

t	Beg. bal.	PMT	Interest	Amort.	End. bal.
1	B_1	P_1	$C_1 = iB_1$	$A_1 = P_1 - iB_1$	$B_2 = (1+i)B_1 - P_1$
2	B_2	P_2	$C_2 = iB_2$	$A_2 = P_2 - iB_2$	$B_3 = (1+i)B_2 - P_2$
3	B_3	P_3	$C_3 = iB_3$	$A_3 = P_3 - iB_3$	$B_4 = (1+i)B_3 - P_3$
4	B_4	P_4	$C_4 = iB_4$	$A_4 = P_4 - iB_4$	$B_5 = (1+i)B_4 - P_4$
….	…	…	…	…	…
t-1	B_{t-1}	P_{t-1}	$C_{t-1} = iB_{t-1}$	$A_{t-1} = P_{t-1} - iB_{t-1}$	$B_t = (1+i)B_{t-1} - P_{t-1}$
t	B_t	P_t	$C_t = iB_t$	$A_t = P_t - iB_t$	$B_{t+1} = (1+i)B_t - P_t$
t+1	B_{t+1}	P_{t+1}	$C_{t+1} = iB_{t+1}$	$A_{t+1} = P_{t+1} - iB_{t+1}$	$B_{t+2} = (1+i)B_{t+1} - P_{t+1}$
…	…	…	…	…	…
n	B_n	$P_n = B_n + iB_n$	$C_n = iB_n$	$A_n = B_n$	$B_{n+1} = 0$

t=time subscript, Beg. bal.=beginning balance, PMT=payment, interest=interest costs,
End. bal.=ending balance

B. An n period i×100% ordinary term loan

An ordinary term loan requires all payments P_ts to be fixed to a level constant P.
It implies that its level payment P always exceeds the interest cost iB_t for any t so the initial loan balance B_1 can be gradually reduced over the term n toward 0.

Set the P_t of the equation in definition 1 equal to P for all t, 1≤ t≤n, as follows:

$$B_1 = \frac{P}{(1+i)^1} + \frac{P}{(1+i)^2} + ... \frac{P}{(1+i)^n}. \tag{2}$$

Solving the above for P, the following result is in order:

$$P = \frac{\left[i(1+i)^n\right]}{\left[(1+i)^n - 1\right]} B_1. \tag{3}$$

The payment P is an important quantity because any entry of its amortization schedule can be generated, once its loan balance and its payment P for each row are known. See its amortization schedule below.

TABLE 2
THE AMORTIZATION SCHEDULE OF AN N PERIOD I × 100% ORDINARY TERM LOAN

t	Beg bal	PMT	Interest	Amort.	End bal.
1	B_1	P	iB_1	$P-iB_1$	$(1+i)B_1-P$
2	B_2	P	iB_2	$P-iB_2$	$(1+i)B_2-P$
…	…	…	…	…	…
t	B_t	P	iB_t	$P-iB_t$	$(1+i)B_t-P$
…	…	…	…	…	…
n-1		P	iB_{n-1}	$P-iB_{n-1}$	$(1+i)B_{n-1}-P$
n	B_n	P	iB_n	$P-iB_n=B_n$	0

t=time subscript, beg. bal.=beginning balance, PMT=payment, interest=interest costs, end. bal.=ending balance

$$P=\frac{\left[i(1+i)^n\right]}{\left[(1+i)^n-1\right]}B_1$$

C. A d period deferred n period i×100% term loan

For simplicity, let us agree that d is a natural number much smaller than the term n of the loan.[7] The amortization schedule of this loan is presented in Table 3.

Inspect Table 3. There is no payment to be made during the initial d periods: =0 for t, 1≤ t≤d. The interest payment for t=1 is . Let us determine the 2nd loan balance . Subtracting the latter for the P, the amortization is - . Thus, the 2nd balance is . The remainder of the row can be generated in a similar way. The loan balance of the third row, B_3 is $B_2(1+i)^1$, which equals $B_1(1+i)^2$. By induction, the t-th loan balance B_t can be expressed as the following:

$$B_t=B_1(1+i)^{t-1} \tag{4}$$

for t, $1\leq t\leq d$.
Substitute d+1 into t of (4). The loan balance B_{d+1} is the following:

$$B_{d+1}=B_1(1+i)^d\ . \tag{5}$$

The latter B_{d+1} will be repaid by a level payment P in the next n months. This requires the following equation to hold:

$$B_1(1+i)^d=\frac{P}{(1+i)^1}+\frac{P}{(1+i)^2}+...+\frac{P}{(1+i)^n}\ . \tag{6}$$

Solving (6) for the level payment P,

$$P = B_1(1+i)^d \frac{\left[i(1+i)^n\right]}{\left[(1+i)^n - 1\right]}. \tag{7}$$

Once the level payment P is determined, the remaining n rows of the amortization schedule can be readily generated in a way similar to that of an ordinary term loan as shown in Table 3.

TABLE 3
THE AMORTIZATION SCHEDULE OF A D PERIOD DEFERRED N PERIOD I $\times$100% TERM LOAN

t	Beg. bal.	PMT	Interest	Amort.	End. bal.
1	B_1	0	$iB_1(1+i)^0$	$A_1 = -B_1$	$(1+i)^1 B_1$
2	B_2	0	$iB_1(1+i)^1$	$A_2 = -B_2$	$(1+i)^2 B_1$
…	…	0	…	…	…
d-1	B_{d-1}	0	$iB_1(1+i)^{d-1}$	$A_{d-1} = -iB_1(1+i)^{d-1}$	$(1+i)^d B_1$
d	B_d	P	iB_d	$A_n = P - iB_d$	$(1+i)B_d - P$
…	…	…	…	…	…
d+t	B_{d+t}	P	iB_{d+t}	$A_{d+t} = P - iB_{d+t}$	$(1+i)B_{d+t} - P$
…	…	…	…	…	…
d+n	B_{d+n}	P	iB_{d+n}	$A_n = P - iB_{d+n}$	0

t=time subscript, beg. bal.=beginning balance, PMT=payment, interest=interest costs, end. bal.=ending balance,

$$P = \frac{\left[i(1+i)^n\right]}{\left[(1+i)^n - 1\right]}(1+i)^d B_1$$

D. An n period $i \times 100\%$ annuity due

This is the most troublesome case for this work because the payment P for the first period is made at t=0 rather than at the end of each period. There exists no amortization schedule for this loan. For an annuity due to be a member of the G-term loan, it is necessary to create its amortization schedule for this loan.

To this end, let F denote the loan face value from the annuity due. Let P denote its level payment. Let B_1 be the loan proceeds from an annuity due, which a borrower receives from a lender at the beginning t=0. A lender must pre-collect the first payment P from the loan face value F of the annuity due at t=0 so that the following relationship must hold:

$$B_1 = F - P. \tag{8}$$

The second payment P must be made precisely at time point t=1, which is the beginning of the second period. Since time t is a continuum, the last moment of the previous period can be viewed as the first moment of the next period. Under this interpretation, there are n-1 payments left to be made after the borrower receives the loan proceeds from the lender at time point t=0. The annuity due will be considered to be paid off at the beginning of the n-th period which is equal to the end of the n-1-th period.

The level payment P must be determined so that the present value of n-1 payments of P discounted at the interest rate of i equals the loan balance , for the first period which the borrower receives from the lender as follows:

$$B_1 = \frac{P}{(1+i)^1} + \frac{P}{(1+i)^2} + \ldots + \frac{P}{(1+i)^{n-1}} \tag{9}$$

Solving the above for P, the following results:

$$P = \frac{\left[i(1+i)^{n-1}\right]}{\left[(1+i)^{n-1} - 1\right]} B_1 . \tag{10}$$

Substituting the P of (10) above into (8), the face loan value F of the annuity due can be precisely determined as follows:

$$F = \frac{\left[(1+i)^n - 1\right]}{\left[(1+i)^{n-1} - 1\right]} B_1 . \tag{11}$$

In light of the preceding discussion, the following definition is in order:

Definition 2:

The amortization schedule of an n period i×100% annuity due is by stipulation the amortization schedule of an n-1 i × 100% ordinary term loan in this work♦

With this definition, it is possible to state that an annuity due has its own amortization schedule. Table 4 presents the amortization schedule for the annuity due.

TABLE 4
THE AMORTIZATION SCHEDULE OF AN N PERIOD I × 100% ANNUITY DUE

t	Beg. bal.	PMT	Interest	Amort.	End bal.
1	B_1	P	iB_1	$P\text{-}iB_1$	$B_2=(1+i)B_1-P$
2	B_2	P	iB_2	$P\text{-}iB_2$	$B_3=(1+i)B_2-P$
…	…	…	…	…	…
t	B_t	P	iB_t	$P\text{-}iB_t$	$B_{t+1}=(1+i)B_t-P$
…	…	…	…	…	…
n-1	B_{n-1}	P	iB_{n-1}	$P\text{-}iB_{n-1}$	$(1+i)B_{n-1}-P=0$

t=time subscript, beg. bal.=beginning balance, PMT=payment, interest=interest costs, end. bal.=ending balance F=loan face value B_1=loan proceeds from an annuity due

$$P=\frac{\left[i(1+i)^{n-1}\right]}{\left[(1+i)^{n-1}-1\right]}B_1 \quad F=\frac{\left[(1+i)^{n}-1\right]}{\left[(1+i)^{n-1}-1\right]}B_1$$

E. An n period i × 100% balloon loan

Consider an n period i×100% balloon loan. Let the level payment of a balloon loan be denoted by P. The level payment P of the balloon loan can be found by solving the following:

$$B_1=\frac{P}{(1+i)^1}+\frac{P}{(1+i)^2}+.....+\frac{P}{(1+i)^t}+...+\frac{P}{(1+i)^h} \tag{12}$$

where h is the term of an i×100% ordinary term loan. The following level payment results:

$$P=\frac{\left[i(1+i)^{h}\right]}{\left[(1+i)^{h}-1\right]}B_1 . \tag{13}$$

The h is much greater than the term n of a balloon loan so that the level payment becomes low. A borrower must pay the level payment P until the last year n. On the last year n, he has to pay off the outstanding balance of the loan and the last interest. The balloon payment P_n must be set so that the present value of n-1 level payment of P plus the present value of the balloon payment P_n discounted at the interest rate i is equal to the loan amount B_1 as implicit in the equation (1) of definition 1 as follows:

$$B_1=\frac{P}{(1+i)^1}+\frac{P}{(1+i)^2}+...+\frac{P}{(1+i)^{n-1}}+\frac{P_n}{(1+i)^n} . \tag{14}$$

Summing the geometric series on the right-hand side, and solving the resulting equation for P, the following results:

$$P_n = \frac{\left[(1+i)^{h+1} - (1+i)^n\right]}{\left[(1+i)^h - 1\right]} B_1 \, . \tag{15}$$

The amortization schedule of the balloon loan is presented in Table 5.

TABLE 5
THE AMORTIZATION SCHEDULE OF AN N PERIOD I×100% BALLOON LOAN

t	Beg bal	PMT	Interest	Amort.	End bal.
1	B_1	P	B_n	P-iB_1	$(1+i)B_1 - P = B_2$
2	B_2	P	iB_2	P-iB_2	$(1+i)B_2 - P = B_3$
…	…	…	…	…	…
t	B_t	P	iB_t	P-iB_t	$(1+i)B_t - P = B_{t+1}$
…	…	…	…	…	…
n-1	B_{n-1}	P	iB_{n-1}	P-iB_{n-1}	$(1+i)B_{n-1} - P = B_n$
n	B_n	$P_n = B_n + iB_n$	iB_n	P_n-i$B_n = B_n$	$B_{n+1} = 0$

t=time subscript, beg. bal.=beginning balance, PMT=payment, interest=interest costs, end. bal.=ending balance, P=level payment, P_n=balloon payment

F. An n period i × 100% interest-only loan

This is essentially identical to that of a balloon loan when the term n of a balloon is equal to h and the level payment P is set equal to the interest payment iB_1. The balloon payment P_n is nothing but the face value of the loan plus the interest payment. An interest-only loan is often used in real estate transactions when the term n is not long. It is also referred to as a bond in investment if the term n becomes long such as 20 years. See Table 6 below.

TABLE 6
THE AMORTIZATION SCHEDULE OF AN IN PERIOD I × 100% INTEREST-ONLY LOAN

t	Beg bal	PMT	Interest	Amort.	End bal.
1	B_1	iB_1	iB_1	0	B_1
2	B_1	iB_1	iB_1	0	B_1
…	…	…	…	…	…
t	B_1	iB_1	iB_1	0	B_1
…	…	…	…	…	…
n-1	B_1	iB_1	iB_1	0	B_1
n	B_1	$(1+i)B_1$	iB_1	B_1	0

t=time subscript, beg. bal.=beginning balance, PMT=payment, interest=interest costs, end. bal.=ending balance,

G. An n period $i \times 100\%$ bullet loan

The amortization schedule of this loan can be readily constructed from the d period deferred n period term loan by setting n equal to 0. The upper half of the amortization schedule becomes the amortization schedule of the bullet loan with a minor modification on the d-th row. See Table 7 below:

TABLE 7
THE AMORTIZATION SCHEDULE OF AN N PERIOD I × 100% BULLET LOAN

t	Beg. bal.	PMT	Interest	Amort.	End. bal.
1	B_1	0	$iB_1(1+i)^0$	$-iB_1(1+i)^0$	$(1+i)^1 B_1$
2	$(1+i)^1 B_1$	0	$iB_1(1+i)^1$	$-iB_1(1+i)^1$	$(1+i)^2 B_1$
…	…	0	…	…	…
n-1	$(1+i)^{n-2} B_1$	0	$i(1+i)^{n-2} B_1$	$-i(1+i)^{n-2} B_1$	$(1+i)^{n-1} B_1$
n	$(1+i)^{n-1} B_1$	$(1+i)^n B_1$	$i(1+i)^{n-1} B_1$	$(1+i)^{n-1} B_1$	0

t=time subscript, beg. bal.=beginning balance, PMT=payment, interest=interest costs, end. bal.=ending balance,

It has been demonstrated that all G-term loans always have their own amortization schedule. This property will be used as the key premise in asserting the validity of the new theory of term loans on the section since the main theorem to be introduced next will be proved by using the amortization schedule for a G-term loan.

IV. A NEW THEORY OF G-TERM LOANS

Let us state the main theorem of this work and establish its validity next.

Theorem 1:
For any t, $2 \le t \le n$

$$B_t = (1+i)^{t-1} B_1 - (1+i)^{t-2} P_1 - (1+i)^{t-3} P_2 - \dots - (1+i)^1 P_{t-2} - (1+i)^0 P_{t-1} \tag{16}$$

and

$$B_t = \frac{P_t}{(1+i)^1} + \frac{P_{t+1}}{(1+i)^2} + \dots + \frac{P_{n-1}}{(1+i)^{(n-1)-(t-1)}} + \frac{P_n}{(1+i)^{n-(t-1)}} \blacklozenge \tag{17}$$

A. Proof of the t-th loan balance B_t of (16):

In proving the validity of the t-th balance B_t of (16) for any natural t, a proof technique known as mathematical induction will be utilized. The procedure involves establishing the validity of the B_t for some initial cases, for instance, t= 2, 3, and 4 in this case. Then, hypothesize the validity of the B_t of (16) for an arbitrary t, which is technically referred to as the inductive hypothesis. The critical step of proof is to show that the inductive hypothesis holds for the period, t+1. Then, the hypothesis holds true for any natural t by mathematical induction.

Observe the amortization schedule of a G-term loan in Table 1. From the ending balance column of row 1 of the amortization schedule of the G-term loan,

$$B_2 = (1+i)^1 B_1 - (1+i)^0 P_1 \,. \tag{18}$$

From the ending balance column of row 2 of the amortization schedule,

$$B_3 = (1+i)^1 B_2 - (1+i)^0 P_2 \,. \tag{19}$$

Substitute the B_2 of (18) into the B_2 of (19),

$$B_3 = (1+i)^2 B_1 - (1+i)^1 P_1 - (1+i)^0 P_2 \,. \tag{20}$$

From the ending balance column of row 3 of the amortization schedule,

$$B_4 = (1+i) B_3 - (1+i)^0 P_3 \,. \tag{21}$$

Substitute the B_3 of (20) into the B_3 of (21). Then,

$$B_4 = (1+i)^3 B_1 - (1+i)^2 P_1 - (1+i)^1 P_2 - (1+i)^0 P_3 \,. \tag{22}$$

In light of (20) for t=3 and (22) for t=4, the inductive hypothesis is that

$$B_t = (1+i)^{t-1} B_1 - (1+i)^{t-2} P_1 - (1+i)^{t-3} P_2 - \ldots - (1+i)^0 P_{t-1} \,. \tag{23}$$

The key step of proof will be presented next. From the ending balance column of row t of the G-term loan amortization schedule in Table 1,

$$B_{t+1} = (1+i) B_t - (1+i)^0 P_t \,. \tag{24}$$

Substitute the of (23) into the of (24) on the right-hand side as follows:

$$B_{t+1} = (1+i)\left[(1+i)^{t-1} B_1 - (1+i)^{t-2} P_1 - (1+i)^{t-3} P_2 - \ldots - (1+i)^0 P_{t-1}\right] - P_t \tag{25}$$

which is equal to the following:

$$B_{t+1} = (1+t)^t B_1 - (1+i)^{t-1} P_1 - (1+i)^{t-2} P_2 - \ldots - (1+i)^1 P_{t-1} - (1+i)^0 P_t \,. \tag{26}$$

Replace the t+1 of equation (26) by the Greek letter τ. Then, the following result is in order:

$$B_\tau = (1+i)^{\tau-1} B_1 - (1+i)^{\tau-2} P_1 - (1+i)^{\tau-3} P_2 - \ldots - (1+i)^1 P_{\tau-2} - (1+i)^0 P_{\tau-1} \,. \tag{27}$$

The (27) above is mathematically equivalent to the inductive hypothesis of (20). Thus, it has been shown that the t-th balance formula of the inductive hypothesis holds true for any natural t, t=2,3,…,n by mathematical induction♦

B. Proof of the t-th loan balance B_t of the (17):

Multiply (1) of Definition 1 by $(1+i)^{t-1}$. The following result is in order:

$$(1+i)^{t-1}B_1=(1+i)^{t-2}P_1+(1+i)^{t-3}P_2+\ldots+(1+i)^0P_{t-1}+\frac{P_t}{(1+i)^1}+\frac{P_{t+1}}{(1+i)^2}+\ldots+\frac{P_n}{(1+i)^{n-(t-1)}}\,. \tag{28}$$

Transfer the first summation term to the left.

$$\begin{aligned}(1+i)^{t-1}B_1-(1+i)^{t-2}P_1-(1+i)^{t-3}P_2-(1+i)^{t-4}P_3-\ldots-(1+i)^1P_{t-2}-(1+i)^0P_{t-1}\\ =\frac{P_t}{(1+i)^1}+\frac{P_{t+1}}{(1+i)^2}+\ldots+\frac{P_{n-1}}{(1+i)^{(n-1)-(t-1)}}+\frac{P_n}{(1+i)^{n-(t-1)}}\end{aligned} \tag{29}$$

In light of the (16), the left-hand side of the above equation is nothing but the t-th loan balance, itself. Thus, the right-hand side of the (29) above must be also as follows:

$$B_t=\frac{P_t}{(1+i)^1}+\frac{P_{t+1}}{(1+i)^2}+\ldots+\frac{P_{n-1}}{(1+i)^{n-1-(t-1)}}+\frac{P_n}{(1+i)^{n-(t-1)}}\ \blacklozenge \tag{30}$$

Now that the validity of the new theory is rigorously established, a stage is set for deriving the algebraic formulae.

IV. LOAN BALANCE FORMULAE

Consider an n period i×100% ordinary term loan. The t-th balance formula B_t will be derived from the (16) of theorem 1 first. Then, the same formula B_t is again derived, this time from the (17) of theorem 1.

Consider an n period i ×100% term loan. Consider the t-th loan balance B_t. Set all P_t s of the (16) of theorem 1 equal to P. The (16) becomes the following:

$$B_t=B_1(1+i)^t-(1+i)^{t-2}P-(1+i)^{t-3}P-\ldots-(1+i)^0P. \tag{31}$$

Summing up the future value of t-1 level payments of P in (31), the following results:

$$B_t=(1+i)^{t-1}B_1-\frac{(1+i)^{t-1}-1}{i}P. \tag{32}$$

Recall the level payment P of an ordinary term loan was presented in (3). Substitute the latter into (32). Then,

$$B_t=(1+i)^{t-1}B_1-\frac{\left[i(1+i)^n\right]}{\left[(1+i)^n-1\right]}B_1\frac{\left[(1+i)^{t-1}-1\right]}{i}. \tag{33}$$

Combine the two terms on the right-hand side into one, and the following loan balance formula is derived:

Theorem 2:

$$B_t = B_1 \frac{\left[(1+i)^n - (1+i)^{(t-1)}\right]}{\left[(1+i)^n - 1\right]} \blacklozenge \tag{34}$$

Let us derive the loan balance formula B_t by means of the (17) of theorem 1. Notice that t-1 level payments of P have been paid prior to the period t. Since there are n level payments of P in total, there remains n-(t-1) level payments of P left. Thus, the (17) becomes the following equation.

$$B_t = \frac{P}{(1+i)^t} + \frac{P}{(1+i)^{t+1}} + \ldots + \frac{P}{(1+i)^{n-1-(t-1)}} + \frac{P}{(1+i)^{n-(t-1)}}. \tag{35}$$

Summing the series,

$$B_t = P \frac{\left[(1+i)^{n-(t-1)} - 1\right]}{\left[i(1+i)^{n-(t-1)}\right]}. \tag{36}$$

Substitute the P of (3) into the above. The following results:

$$B_t = B_1 \frac{\left[i(1+i)^n\right]}{\left[(1+i)^n - 1\right]} \frac{\left[(1+i)^{n-(t-1)} - 1\right]}{\left[i(1+i)^{n-(t-1)}\right]}. \tag{37}$$

The latter simplifies into the right-hand side of (34) of theorem 2. It has been clearly verified that both (16) and (17) result in the identical outcome♦

The t-th balance formulae of a deferred term loan, an annuity due, and a balloon loan can be readily derived from theorem 1. The loan balance formulae of a bullet loan and that of an interest-only loan are available from their amortization tables 6 and 7. Table 8 is the summary tables for all loan balance formulae.

TABLE 8
N THE LOAN BALANCE FORMULAE

Id	Loan types	the t-th loan balance formulae
1	An ordinary term loan B_t=	$B_1 \frac{\left[(1+i)^n - (1+i)^{t-1}\right]}{\left[(1+i)^n - 1\right]}$
2	A deferred term loan B_t=	$B_1\ (1+i)^{t-1}$ for any t, $1 \le t \le d$ $B_1 (1+i)^d \frac{\left[(1+i)^n - (1+i)^{(t-d)-1}\right]}{\left[(1+i)^n - 1\right]}$ for any t, $d+1 \le t \le d+n$
3	An annuity due B_t=	$B_1 \frac{\left[(1+i)^{n-1} - (1+i)^{t-1}\right]}{\left[(1+i)^n - 1\right]}$
4	A balloon loans B_t	$B_1 \frac{\left[(1+i)^h - (1+i)^{t-1}\right]}{\left[(1+i)^h - 1\right]}$
5	A bullet loan B_t	$B_1 (1+i)^{t-1}$
6	An interest-only loan B_t	$B_t = B_1$

B_t=loan balance for the t-th month
B_1=the loan balance for t=1
F=the face value of an annuity due to apply for
i=interest rate per month
h=term of a loan to determine a monthly payment on a balloon loan
d=number of months deferred
n=term of a loan

Now that all loan balance formulae are obtained, let us proceed to the final topic of how the annual interest formulae are derived.

VI. ANNUAL INTEREST FORMULAE

Let us consider a mortgage loan. The term of a loan will be stated in months and years in this section since no generality in terminology appears to be relevant any longer in discussing annual interest tax savings on a mortgage loan. It is necessary to calculate annual interest expenses stipulating that taxes are paid at the end of a year.[8] Let CI(L) denote the cumulative interest expenses for the remaining L months of the first year.

$$CI(L) = \sum_{t=1}^{L} C_t \,. \tag{38}$$

The first task is to compute the annual interest expense for the first year. Define CI(0) to be 0. Assume that the L does not exceed 12. The cumulative interest expense CI(L) for the first L months will be the following:

$$\Delta CI(L,0)=CI(L)-CI(0). \tag{39}$$

If L is exactly 12, the above is the annual interest expense for the first year. Suppose that this mortgage is issued on the month 12-L where L is less 12. The difference above is the annual interest expense for the remaining L months for the first year.

Consider another year where L is the last month of the previous year and M the ending month of this year. Let ΔCI(M,L) denote interest accrued from the L+1-th month to the M-th month. M is at least greater than L+1 where L represents the first month of a year and M the last month of the same year. Subtract the cumulative interest CI(L) for L months from the cumulative interest CI(M) for M months. Then,

$$\Delta CI(M,L)=CI(M)-CI(L). \tag{40}$$

The above difference is the annual interest expense for this particular year.

For example, a mortgage loan is issued on September 1. Then L is 12-9=4. The total interest expenses accrued from September to December is the first-year interest accrued. The first year annual interest expense is obtained by the following:

$$\Delta CI(4,0)=CI(4)-CI(0) \tag{41}$$

where

$$CI(4)=\sum_{t=1}^{4} C_t \tag{42}$$

and

$$CI(0)=0. \tag{43}$$

The annual interest expenses next year should be given by the following:

$$\Delta CI(16,4)=CI(16)-CI(4) \tag{44}$$

where

$$CI(16)=\sum_{t=1}^{16} C_t \tag{45}$$

and

$$CI(4)=\sum_{t=1}^{4} C_t. \tag{46}$$

The difference ΔCI(16,4) is the second-year annual interest expense.

Consider an ordinary term loan. Let us derive CI(L) in order to illustrate how the annual interest formula is derived. In light of (34) of theorem 2,

$$CI(L)=\sum_{t=1}^{n} i\frac{\left[(1+i)^{n}-(1+i)^{(t-1)}\right]}{\left[(1+i)^{n}-1\right]}. \tag{47}$$

Summing up the geometric series of (47) on the right-hand side:

$$CI(L)=B_1\frac{\left[i(1+i)^{L}L-(1+i)^{L}-1\right]}{\left[(1+i)^{n}-1\right]}. \tag{48}$$

Then, in light of (40),

$$\Delta CI(M,L))=\frac{\left[i(1+i)^{n}(M-L)-(1+i)^{M}+(1+i)^{L}\right]}{\left[(1+i)^{n}-1\right]}. \tag{49}$$

The interest formulae for the ordinary term loan, the deferred term loan, the annuity due, and the balloon of G-term loans can be derived in a similar way. The remaining two cases require a separate discussion.

A bullet loan and an interest-only loan are special cases. Consider the former first. The t-th interest cost C_t is $iB_1(1+i)^{t-1}$. Summing up the latter geometric series,

$$\Delta CI(0,L)=B_1\left[(1+i)^{L}-1\right]. \tag{50}$$

which gives the annual interest expenses for the first year. The annual interest expenses for the later years are given by the following:

$$\Delta CI(L,M)=B_1\left[(1+i)^{M}-(1+i)^{L}\right]. \tag{51}$$

The interest-only loan is a simple case since the interest is a level constant i. The annual interest expense for the first year is merely the following:

$$\Delta CI(0,L)=iB_1L. \tag{52}$$

The annual interest expense for any remaining year are given by the following:

$$\Delta CI(M,L)=iB_1(M-L). \tag{53}$$

The M-L is 12 for a complete year. It is less than 12 when the last year is not a complete year.

The following table 9 is the summary table of annual interest expense formulae for the four loans of our primary interest.

TABLE 9
THE ANNUAL INTEREST FORMULAE

Id	Loan types	Annual interest expenses
1	An ordinary term loan	
	$\Delta CI(0,L) =$	$B_1 \frac{\left[i(1+i)^n L - (1+i)^L + 1\right]}{\left[(1+i)^n - 1\right]}$
	$\Delta CI(L,M) =$	$B_1 \frac{\left[i(1+i)^n (M - L) - (1+i)^M + (1+i)^L\right]}{\left[(1+i)^n - 1\right]}$
2	A deferred term loan	
	A deferred term loan $\Delta CI(0,L)$ L>d	$B_1 (1+i)^d \frac{\left[i(1+i)^n (L - d) - (1+i)^{L-d} + 1\right]}{\left[(1+i)^n - 1\right]}$
	$\Delta CI(L,M)$	$B_1 (1+i)^d \frac{\left[i(1+i)^n (M - L) - (1+i)^{L-d} + (1+i)^{M-d}\right]}{\left[(1+i)^n - 1\right]}$
3	An annuity due	
	$\Delta CI(0,L)$	$B_1 \frac{\left[i(1+i)^{n-1} L - (1+i)^L + 1\right]}{\left[(1+i)^n - 1\right]}$
	$\Delta CI(L,M)$	$B_1 \frac{\left[i(1+i)^{n-1} (M - L) - (1+i)^M + (1+i)^L\right]}{\left[(1+i)^n - 1\right]}$
4	A balloon loan	
	$\Delta CI(O,L)$	$B_1 \frac{\left[i(1+i)^h L - (1+i)^L + 1\right]}{\left[(1+i)^h - 1\right]}$
	$\Delta CI(M,L)$	$B_1 \frac{\left[i(1+i)^h (M - L) - (1+i)^M + (1+i)^L\right]}{\left[(1+i)^h - 1\right]}$
5	A bullet loan	
	$\Delta CI(O,L)$	$B_1 \left[(1+i)^L - 1\right]$
	$\Delta CI(M,L)$	$B_1 \left[(1+i)^M - (1+i)^L\right]$
6	An interest-only loan	
	$\Delta CI(0,L)$	$i B_1 L$
	$\Delta CI(M,L)$	$i B_1 (M - L)$

$\Delta CI(0,L)$=cumulative interest expense up to month L from the beginning
$\Delta CI(L,M)$ = interest expenses accrued between month L and month M
h=term of a loan to determine a monthly payment on a balloon loan
d=number of months deferred
n=term of a loan for all except a balloon loa

VII. SUMMARY AND CONCLUDING REMARKS

This paper introduced a new category of term loans named G-term loans, which are nothing but popular term loans used in business. The definition of G-term loans in this work was rigorously formulated and its key property was stated. The paper then presented the main theorem of this work and proved its validity. The loan balance formulae for G-term loans were then derived from the theorem. Finally, the formulae to derive annual interest expenses were in turn derived from the loan balance formulae.

The algebraic formulae are a powerful tool for computing annual interest expenses, and hence tax savings on the latter. They are needed in making financial decisions, especially in refinancing analysis as well as in other financial decisions such as lease or buy bond refunding decisions, and capital budgeting analysis. They are of great value for finance students learning how to conduct refinancing analysis in classroom settings. The reason is that nearly all finance students today are competent Excel users and they can store the new formulae to compute annual interest expenses in cells of an Excel worksheet without any problem. They can develop an Excel worksheet to conduct refinancing analysis under the algebraic formula approach. Further, they can design an Excel model to conduct a sensitivity and simulation analysis of refinancing analysis on their laptop computers. In this regard, it should be kept in mind that many of these students are future financial decision makers who will utilize the new formulae in practice. Finally it is noted in passing that the algebraic formulae will be also needed by computer programmers engaged in designing efficient business software in these areas.

There are several important implications to be stressed. The new formula approach necessitates a re-examination of the traditional pedagogy of teaching interest computation based on an amortization schedule. An amortization schedule will continue to be the best way for students to understand the mechanics of how a loan is gradually paid off over the term of a loan. Hence, it has to be included in the time value of money chapter in introductory finance courses.

However, they should not use the amortization schedule as a computational tool of calculating interest expenses because the amortization schedule method of interest computation is a grossly inefficient way to assess annual interest expenses needed in financial decisions. Further, keep in mind that refinancing analysis has to be typically repeated a number of times in practice since interest and tax rates are often viewed as stochastic and decision makers want a sensitivity and simulation analysis to be performed after inspecting the initial outcome from replacement analysis. Thus, computational burden is bound to become overwhelming under the amortization schedule method.

ENDNOTES

1. To expound on this point, if a loan payment is not sufficient in a certain period to cover its period interest due, the unpaid part of the accrued interest will be added to the outstanding balance of the loan without any penalty. Whereas, when a loan payment exceeds its period interest accrued, the loan balance will be reduced by the amount exceeding the accrued interest expense.
2. Some of these terms are business jargon used by realtors, for instance. A bullet loan in this work is a loan which has no intermediate payment and whose face value and accrued interest must be paid at maturity. An interest-only loan pays its interest every period and the interest plus the principle at maturity. It is similar to a bond.
3. The book seems to be also a commercial success in light of the fact that the second edition appeared recently. See Benninga (2011).
4. In our view, students must experience at least once in an introductory finance course how to generate an amortization schedule, though it should no longer be used as a primary tool to calculate accrued interest, for instance mortgage refinancing analysis.
5. The period here is either a year or a month. In refinancing a mortgage, a payment has to be paid monthly. So, a period t is a month t.
6. The diamond ♦ is used as a terminator for emphasis.

7. The deferment period d could be shorter than 1. This often occurs when the closing date of a mortgage is, for example, on the 28th of the month and a lender wants to collect the mortgage payment on the first day of a month. A common practice is to increase the amount of the loan by adding the interest for the extra few days to the principal so that the mortgage payment is due exactly at the end of each month. This method is often used by lenders. For this reason, a deferred term loan is quite common.
8. This is just for generality. In fact, in the United States, individual taxes are due in April.

REFERENCES

Benninga, Simon. (2005). *Principles of Finance with Excel*, 1st ed., (Oxford University Press: New York, NY).

_______________(2008). *Financial Modeling* 3rd ed., (MIT Press: Cambridge, MA)

_______________(2011). *Principles of Finance with Excel*, 3rd ed., (Oxford University Press: New York, NY)

Cagle, Julie, A. B., Philip W. Glasgo, and David C. Hyland. (2010). "Spreadsheets: Do They Improve Student Learning in the Introductory Finance Course," *Journal of Financial Education*, 36 (3/4) (Fall/Winter) pp. 35-52.

Chen, Richard. (1997). "A Refinancing Decision Model Using Spreadsheet Software," *National Public Accountant.* 42(2) pp. 44-48.

Harris, Robert, S., and John J. Pringle. (1985). "Risk-Adjusted Discount Rate—Extensions from the Average-Risk Case," *Journal of Financial Research,* 8(3), pp. 237-244.

Hull, Robert W. and Fred J. Ebeid. (1991). "Amortized Loans: An Alternative Approach to Determining Selected Loan Balances, Equity Values, and Interest Amounts," *The Proceedings of the Midwest Business Association, in Finance and Insurance*, (Spring), Chicago Il. pp. 78-85.

Jackson, Mary and Mike Staunton. (2001). *Advanced Modeling in Finance Using Excel and VBA*, (John Wiley and Sons: Hoboken, NJ)

Keown, Arthur J., John D. Martin, J. William Petty, David F. Scott, Jr. (2010). *Financial Management: Principles and Applications*, 10th ed., (Prentice Hall: Upper Saddle River, NJ).

MacDougall, Shelley L., and Scott B. Follows. (2006). "Model-building in Excel as Pedagogy for the Fundamentals of Finance," *Journal of Financial Education*, 32 (Fall) pp. 37-54.

Matsumoto, K, John Monro, Michael Chang, and Dion Douws. (2012). "An Excel Model of Mortgage Refunding Decisions for Sensitivity Analysis and Simulation." The paper was accepted for publication in the Journal of Accounting and Finance, North American Business Press. An earlier version of the paper is available at the FMA 2011 Annual Meeting website.

Whitworth, Jeff. (2010). "Integrating Excel into the Principles of Finance Course: Financial Statement Analysis and Time Value of Money," *Journal of Financial Modeling and Financial Technology*, 1(1) pp. 1-16.