# Bayesian Design of Inventory Systems to Minimize Expected Operating Costs

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*Inventory management requires optimal determination of (a) "when to order", and (b) "how much to order". One also needs here two answers—what should be stocked initially, and should one adjust (a) and (b) as time progresses? This paper uses the Bayesian approach to answer these. It uses the classical (s, Q) model and a heuristic search in the unstructured decision space. It finds that one can make a decent start by this approach—and stay the course nearly optimally—while upholding a target service level and minimizing cost/unit time. Study finds high value in continually updating (a) and (b).*

## INVENTORY CONTROL THEORY

Inventories are a common insurance against uncertainties impacting most production or service operations ranging from blood banks (plasma, bags and syringes), auto plants (equipment, parts and finished vehicles), hospitals (beds, operating theaters, specialists and drugs), book stores (books and stationeries), supermarkets and stores (household and consumer goods), restaurants (supplies and semi-finished recipes), banks (currency, tradable assets, forms, etc.), project sites (labor, equipment and materials), etc. However, inventories behave characteristically like money placed in a drawer—not producing any "return", and even forcing a borrowing cost (Jensen and Bard 2003). On the positive side, inventories help out in breakdowns or crisis, and generally improve customer service in parts and goods transactions. A balance here may be struck by optimally determining (a) "when to order", and (b) "how much to order". Uncertainties are particularly high for a new business entering an established market. One needs here two answers—how much should the organization stock initially, and should it adjust decisions (a) and (b) as time progresses?

This paper re-visits the Bayesian approach to test its efficacy in answering these two questions.

Much has been researched and reported in the past decades on the optimal way of planning and managing inventories, recent ideas coming in the form of JIT, Lean Production, and the Theory of Constraint's throughput accounting way. Mathematical modeling engaging various types of costs and management objectives has been the broad approach to help evolve policies and decision guidelines. Authoritative references here include Prabhu (1965), Silver et al. (1998), Jensen and Bard (2003), Winston (2003) and several others. Broadly, based on the environment in which inventories are created and managed, two large classes now exist, namely, the constant or deterministic demand rate condition, and the stochastic demand condition when demand fluctuates randomly, following some statistical distribution. As special conditions emerge due to challenges in supply chain management, much scope yet remains to help maximize the utility of strategic inventories while the external world continues to be unpredictable.

This paper is organized as follows. This section recalls the salient approaches to handle inventories optimally. Section 2 summarizes the features of the (*s, Q*) model, an effective approach to minimize costs as well as sustain customer service when demand is random, but controlled by a stationary distribution. Section 3 reports inferences drawn from certain computational experiments conducted with this model. Section 4 introduces the Bayesian approach of continually inferring the true characteristics of demand, and the gradual reduction of uncertainty about the true demand. Section 5 incorporates Bayesian learning into the challenge of keeping the operating values of the decision parameters *s*, the reorder point, and *Q*, the order quantity near optimal. This would keep the total expected inventory management cost minimum as time progresses. Section 6 lists the conclusions.

## IDEALIZATION OF PRODUCT OR SERVICE DEMAND

Inventory modeling, now incorporated into the broad collection of mathematical methods to apply scientific approaches to decision making in stock management, attempts to idealize several aspects of the reality of stocking materials, parts and goods—to make the solution tractable. Simplifications are done, for instance, when one assumes that demand is constant and deterministic. The other extreme is when not only it is random, but even nonstationary (Graves 1999). To produce tractable solutions a large collection of approaches have been proposed (Silver 1981, Winston 2003). This body of knowledge may be divided into the nature of the demand (deterministic and stochastic), the decision period, the nature of the product involved, as well as other aspects. For the purpose at hand, demand is assumed to be stochastic, continuing indefinitely over multiple periods. But broadly, analysis has been split into two classes based on the nature of demand fluctuation as follows.

### Deterministic Demand Condition

One way to deal with the chaotic behavior of market demand is to take the first stab—assume that products are demanded at a *uniform* (constant, = *a*) rate as time progresses. Indeed this was the basis for developing perhaps the first quantitative representation of the inventory manager's decision space, leading to the celebrated EOQ model of inventories. The object here was to set up cost models incorporating the two earlier stated questions (a) and (b)—when to order new stock, and how much to order—to sustain the business. Ordering cost, Setup cost, product cost, holding cost, shortage (back order) cost, etc. were identified as the relevant constituents, leading to the optimal (minimum cost) value for the order quantity *Q*. This optimal order quantity is called the *Economic Order Quantity* (EOQ) as it helps minimize the total cost comprising the setup cost, product cost and holding cost. The correct time between the placement of orders here is *Q/a*, which avoids excess inventory piling up or stock outs due to late receipts. By far, devised some 70 years back, the EOQ model remains the backbone of many "modern" inventory management systems and ERP packages, even if it is known that demand itself is rarely constant—it fluctuates with time, seen frequently to follow the Poisson or other similar distribution (Tersine 1994).

### Stochastic Demand Condition

When uncertainty dominates demand conditions, it becomes difficult to determine how much to order and when to order, so as to keep customer service level sufficiently high while also minimizing the total expected operating cost per unit time. When the quantity ordered is too small, one may lose sales. On the other hand, if the order size is too high, sold quantities many not consume the stock completely and the excess may remain for considerable time tying up working capital, space, etc. and entailing housekeeping, insurance, etc. Researchers have addressed this situation using probabilities. Thus several well-formulated models and their solutions have been developed to address what we call the *stochastic demand* case. Prominent are the single period stochastic model, the (*s, S*) model, and the (*s, Q*) model, descriptions being given in Silver et al. (1998), Jensen and Bard (2003) and Winston (2003).

It is argued by Murray and Silver (1966) that initially one would have a great uncertainty concerning the sales potential of an item but would have the opportunity to develop a better feel for this uncertainty as actual sales become known. *Learning*, in the decision theoretic sense, is the process of basing one's initial decisions on an informed or inspired guess to start one's business, and subsequently updating that initial guess by some rational logic—to assure optimality of future decisions. Various approaches have been proposed to assist one in structured learning, good examples being the construction of econometric models, Bayesian models in management science, and Neural Nets. Econometric models generally use historical data to produce statistical estimates of such quantities as product or service demand, while neural networks use a training-validation-testing approach to help learning to occur about the future possible course of a datum of interest (here demand).

The Bayesian approach of learning or adaptation to help develop a better sales forecast appears to have been used first by Murray and Silver (1966). They used Bayes' Rule to adaptively change the distribution of sales and Dynamic Programming to show how decisions may be thus improved. Other studies that have used adaptation-aimed learning in inventory design/operation problems include Sharma et al. (2001), Huh et al. (2009), Huh et al. (2010), and others. Few however have directly addressed the assessment of the potential benefits of Bayesian adaptation/learning of the stochastic demand condition. The exception appears to be the work of Azoury and Miller (1994), who provide a comparison of the optimal ordering levels of Bayesian and Non-Bayesian *n*-period nondepletive inventory models. They show that the quantity ordered under the non-Bayesian policy would be greater than or equal to that under a Bayesian policy. Aside from such applications of learning in managing inventories better, Bayesian methods have been used also to design queuing systems with initially unknown demand (see Bagchi and Cunningham (1972) and Morales et al. (2005) who utilized posterior probabilities of arrivals with the assumption of stationarity of the distribution imposed). For the present study we propose the question,

> *Can the Bayesian learning logic (prior → prior + data → posterior) of observing and updating stochastic information help reduce the total expected cost for operating an inventory management system?*

As detected by Azoury and Miller (1994) for the *n*-period nondepletive inventory model, we anticipate that for the popular (*s*, *Q*) policy, the Bayesian approach would let one see how the successive incorporation of new data would improve decisions (reduce the total expect cost (EC) and/or improve service level) as $Q^*$ and $s^*$ are continually updated with accumulating demand data. Besides, one would also like ask: up to what point such updating would be meaningful? We expect that that answer may depend on the estimated unknown but stationary demand rate for that might require meeting certain minimum sample size. How long should one sample demand (*X*) might depend on the cost of the effect of using non-optimal rather than a near-optimal EC (see Levi et al. 2007). We shall not probe this here.

To study these we use a well-developed stochastic demand inventory model from the literature—the (*s*, *Q*) model. This model incorporates safety stock into the reorder stock level (*s*) and uses an optimum constant quantity *Q* to size release of an order every time the current stock level touches or falls below *s*. The optimum combination of *s* and *Q* minimizes the total expected inventory operating cost per unit time (Silver 2007).

## THE (*s*, *Q*) INVENTORY MODEL

This section is based on the material in Chapter 25 of Jensen and Bard (2003) as it aims to recall the essential and relevant relationships among inventory quantity *z*, the decision variables *s* and *Q*, and costs, to help proceed to the analysis that follows in the rest of this paper. In the present case we assume that only a single item is being stocked and sold whose inventory is to be managed to keep the expected total cost minimum, the total cost comprising holding, replenishment and stock out costs. Winston (2003) also discusses this case.

Ordering too much or too little or at the wrong time can disrupt the optimal control of inventory, an event easily caused by uncertainty or randomness in demand. In such cases the deterministic approach clearly does not minimize the expected total cost.

At some instant of time if inventory level is $z$, then the probability of shortage ($P_s$), the probability of excess ($P_e$), the expected shortage ($E_s$) and expected excess ($E_e$) are, respectively,

$$P_s = P[x > z] = 1 - F(z)$$
$$P_e = P[x \le z] = F(z)$$
$$E_s = \int_z^{\infty} (z - x) f(x) dx \quad (for\ continuous\ demand\ x)$$
$$E_s = \sum_z^{\infty} (x - z) P(x) \quad (for\ discrete\ demand\ x), \quad and$$
$$E_e = z - \mu + E_s$$

The ($s$, $Q$) inventory management policy is adopted in the present paper to serve as the test bed to probe our research question. In this case demand is stochastic. This policy first determines the optimum values ($s^*$ and $Q^*$) for the reorder point ($s$) and the order quantity ($Q$) and then monitors the level of inventory continuously through the repeated execution of order cycles. An order of size $Q^*$ is placed the moment the current inventory level $z$ touches $s^*$. The order (quantity = $Q^*$ ) is received after lead time $L$ and it immediately replenishes the stock.

The optimum parameters $s^*$ and $Q^*$ are found as follows. When $L$ is small compared to the expected time required to exhaust $Q$, only 1 order would be outstanding. (In practice a plant may place multiple orders on a vendor when expediting becomes ineffective, but we do not consider this case here.) An order cycle here is the time between two successive order receipts. If $a$ and $L$ respectively represent the average demand rate and lead time, then the mean demand during lead time is $\mu = aL$. The reorder point being $s$, $P_s$ is $1 - F(s)$, and the system's service level (fraction of demand during lead time that is met) is $1 - P_s = F(s)$. The safety stock (excess stock beyond $\mu$) will thus be $SS = s - \mu$. The general solution for the ($s$, $Q$) policy for this situation has been given by Jensen and Bard (2003) as follows.

If the per SKU unit holding cost is $h$ per unit time, then

Expected holding cost/unit time = $h\left(\frac{Q}{a} + s - \mu\right)$

The time between orders is a random variable with mean of $Q/a$. If the cost of replenishment per order (or the order cost) is $K$ then the expected replenishment cost/unit time is

Expected replenishment cost/unit time = ($Ka/Q$)

If the expected shortage cost per order cycle is $C_s$, then the expected shortage cost/unit time will be $C_s/(Q/a) = C_s a/Q$. The general model for the expected total cost/unit time for the ($s$, $Q$) policy will be

$$EC(s,Q) = h\left(\frac{Q}{a} + s - \mu\right) + \frac{Ka}{Q} + \frac{a}{Q}C_s \qquad (1)$$

Equation (1) gives the general expression for the expected total cost/unit time for an inventory system being operated by the ($s$, $Q$) policy. In order to optimize it we may utilize the two decision variables—$s$, the reorder point, and $Q$, the quantity ordered in each order cycle. Analytically, (1) may be partially differentiated with respect to $s$ and $Q$ and the derivatives equated to zero. Doing this yields two conditions that simultaneously characterize the two optimal values $Q^*$ and $s^*$. These conditions are

$$Q^* = \sqrt{\frac{2a(K + C_s}{h}} \tag{2}$$

and

$$\frac{\partial C_s}{\partial s} = -\frac{hQ}{a} \tag{3}$$

Peterson and Silver (1979) have enlisted several special cases for obtaining the optimal values $Q^*$ and $s^*$. The first case assumes that a constant cost $\pi_1$ is expended whenever a stock out event occurs. This assumption gives us a quick way to evaluate $C_s$—the expected shortage cost/order cycle. This is

$$C_s = \pi_1 P[x > s] = \pi_1 \int_s^\infty f(x)dx = \pi_1[1 - F(s)] \tag{4}$$

Equation (3) may be now utilized since we have $C_s$ expressed in (4) as a function of $s$. Thus

$$\frac{\partial C_s}{\partial s} = -\pi_1 f(s^*) = -\frac{hQ}{a} \tag{5}$$

which gives

$$f(s^*) = \frac{hQ}{\pi_1 a} \tag{6}$$

with

$$C_s = \pi_1(1 - F(s^*) \tag{7}$$

Equation (6) helps link $s^*$ with $Q^*$ via (2).

Note here that seeking a solution to the ($s$, $Q$) policy problem by simultaneously solving (2), (6) and (7) for arbitrary demand distribution F(s) is not trivial.

A variant of the constant cost $\pi_1$ *per stock out event* is a cost $\pi_2$ incurred for *every unit short* in a stock out. Then the expected shortage cost/order cycle will be dependent on how many units are expected to be shorted in each order cycle ($E_s$). Here,

$$E_s = \int_s^\infty (x - s) f(x)dx$$

and therefore $C_s = \pi_2 E_s$. This gives

$$\frac{\partial C_s}{\partial s} = -\pi_2 \int_s^\infty f(x)dx = -\pi_2(1 - F(s)) \tag{8}$$

Combining (3) and (8) one obtains

$$\frac{\partial C_s}{\partial s} = -\pi_2(1 - F(s)) = -\frac{hQ}{a}$$

which for a specified order quantity $Q$ gives the condition for the optimal reorder point $s^*$ as follows.

$$F(s^*) = 1 - \frac{hQ}{\pi_2 a} \quad (9)$$

The optimum decision ($s^*$, $Q^*$) is the combination of $s$ and $Q$ that minimizes EC given by (1).

## VALIDATION OF THE HEURISTIC METHOD OF MINIMIZING EC(*s, Q*) BY MANIPULATING *s* AND *Q*

The Expected Total Cost/unit time (EC) of operating the inventory system given by (1) is a non-linear function of the two decision variables $s$ and $Q$. To optimize (minimize) EC Jensen and Bard (2003) use an iterative procedure (Example 13, Jensen and Bard, 2003). We use this example, which uses a cost $\pi_2$ incurred for *every unit short* in a stock out to compare a heuristic method that Solver® in Excel 2007 uses with the Jensen-Bard iterative procedure. (We made a typographic correction in the Jensen-Bard text. In Example 13 Density φ(.) was changed to CDF Φ(.), to be consistent with Equation 43 in Jensen-Bard.) This example uses the following cost and demand data: Lead time $L$ = 1 week = 0.25 month, Monthly demand $a$ = 100 units/month, normally distributed with variance = 400, giving lead time demand ~ N(25, $10^2$); Holding cost $h$ = \$10/unit-month; Cost expected for every unit short in a stock out event π2 = \$200 per unit backordered; and Order cost K = \$800. Note that in this example, mean demand during lead time $\mu = a/4 = 25$ and the standard deviation of demand during lead time σ = 10.

To find optimal $Q^*$ and $s^*$ together, Jensen and Bard used an iterative procedure that initially assumed $C_s = 0$ and then successively found $Q$, $s$ and the next $C_s$, and then repeated this till $Q$ and $s$ appeared converged. The answers after three iterations were $Q^* = 130.9$ and $s^* = 40.1$. The present work used the Solver analysis option built in Excel® (Solver 2012). A computational Add-in to Excel®, Solver uses a hybrid of classical, metaheuristic and evolutionary algorithms to produce near-optimal solutions. The answers reached by Solver were $Q^* = 127.48$ and $s^* = 40.34$. With this being a satisfactory validation of the Solver-based approach to optimize ($s$, $Q$), we used Solver rather than iteration in this work. In computations we used Equation (2) for $Q^*$, $\Phi(k_s) = 1 - hQ/(a\,\pi2)$, and $C_s = \pi2\ \sigma\ [\varphi(k_s) - k_s\{1 - \Phi(k_s)\}]$. The factor $k_s$ determines the service level, and the optimal reorder point by the relationship $s^* = \mu + k_s\,\sigma$.

## PRELIMINARY INFERENCES FROM COMPUTATIONAL EXPERIMENTS WITH (*s, Q*)

In order to prepare now for investigating whether the Bayesian approach would let one see how the successive incorporation of new data would improve decisions (reduce the total expect cost (EC) or improve service level) as $Q^*$ and $s^*$ are continually updated, we set up the following machinery:

Owing to the non-linear and complex nature of the expressions (1) through (9) we used an orthogonal array experimental computational framework to first determine the sensitivity of the two performances (responses)--% service level and total expected cost. For this we selected two working levels for each of the factors—monthly demand ($a$), holding cost ($h$), shortage cost (π1) and order cost ($K$) as shown in Table 1, and a $L_8$ array (Montgomery 2008). Table 1 and Figures 1 and 2 show the results. The following inferences may be drawn:

- Optimal Order Quantity $Q^*$ is responsive to $a$, $h$ and $K$, but only mildly to shortage cost π1.
- Service level seems to be robust relative to most factors considered in the region of the cost parameters studied. It is closely related to the setting of the reorder point $s^*$.
- Total Expected Cost/unit time is relatively robust with respect to shortage cost per stock out event π1, *but* sensitive to $a$, $h$ and $K$.

These deductions—typically unavailable to the inventory manager—suggest that it would be wise to spend effort in *optimally setting* the reorder point $s^*$ and order quantity $Q^*$ before one sets out to declare the operational policies of an inventory system to cope with stochastic demand. In one sense, this information is similar to the *relative robustness* of the total operating cost/unit time to EOQ for a deterministic inventory system. Limited generalization of such deductions may be attempted in a given cost-demand scenario to assess how accurately the parameters *a*, *h*, *K* and π1 need to be estimated, to assure minimum cost operation of the inventory system. In this study we focus on the factor with perhaps the highest uncertainty—the stochastic nature of *X*, the demand per unit time.

**TABLE 1**
**AN $L_8$ COMPUTATIONAL EXPERIMENT TO UNCOVER THE SENSITIVITY OF SERVICE LEVEL AND TOTAL EXPECTED COST OF AN (*s*, *Q*) INVENTORY SYSTEM**

| Orthogonal Experiment # | *a* Units /month | h | π1 | *K* | $s^*$ | $Q^*$ | %Service Level | Total Expected Cost/unit time |
|---|---|---|---|---|---|---|---|---|
| 1 | 50 | 5 | 500 | 400 | 19.25 | 91 | 97.2 | 488.76 |
| 2 | 50 | 5 | 1000 | 800 | 19.88 | 127.9 | 98.2 | 676.6 |
| 3 | 50 | 10 | 500 | 400 | 18.56 | 64.9 | 95.7 | 709.93 |
| 4 | 50 | 10 | 1000 | 800 | 19.25 | 91 | 97.2 | 977.51 |
| 5 | 100 | 5 | 500 | 800 | 33.61 | 181.2 | 95.74 | 949.29 |
| 6 | 100 | 5 | 1000 | 400 | 36.22 | 128.4 | 98.76 | 698.3 |
| 7 | 100 | 10 | 500 | 800 | 32.51 | 129.1 | 93.35 | 1366.1 |
| 8 | 100 | 10 | 1000 | 400 | 35.41 | 91.5 | 98.13 | 1019.2 |

**FIGURE 1**
**SENSITIVITY OF OPTIMAL REORDER POINT ($s^*$), ORDER QUANTITY ($Q^*$) TO MONTHLY DEMAND RATE AND COSTS**

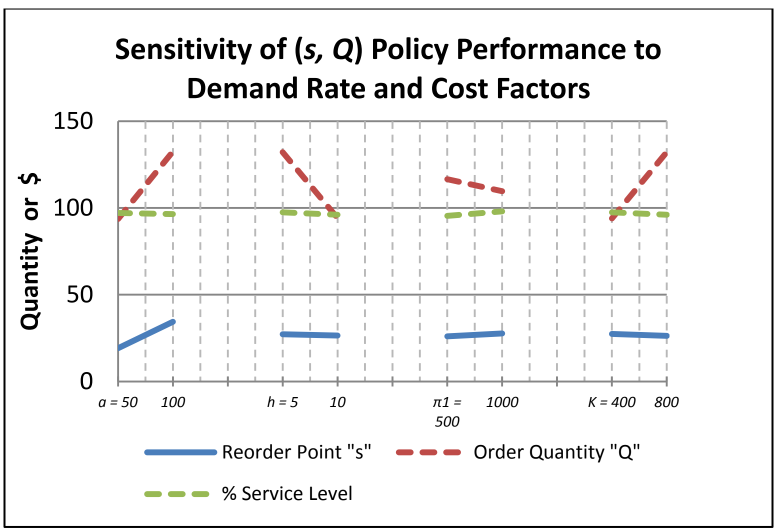
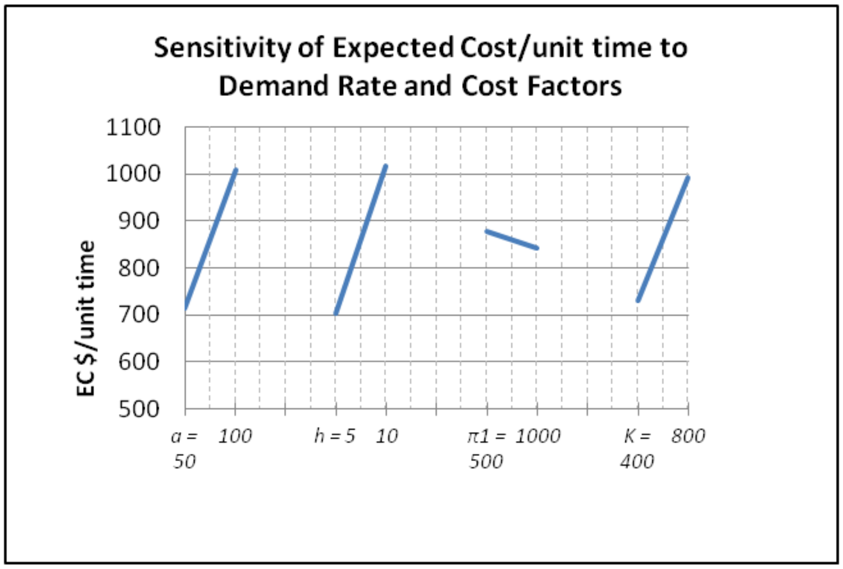


**FIGURE 2**
**SENSITIVITY OF EXPECTED TOTAL COST/UNIT TIME TO MONTHLY DEMAND RATE AND COSTS**




## The Demand Distribution

The demand scenario being stochastic, the demand/unit time is a random variable ($X$), which may be discrete, or continuous. For calculating safety stocks, the majority of stochastic demand models assume the demand during lead time to be normally distributed. Whereas Jensen and Bard (2003) have also used the normal distribution to illustrate the use of the models in determining $s$ and $Q$ in their writing, in the present work we engage the well-justified Poisson distribution (Tersine 1994) for demand. When individual demand events occur independently, the demanded quantity being discrete, one may reasonably assume the underlying distribution to be Poisson. If the average demand rate is $a$ (units of SKD per unit time), then for any time interval t the expected demand is at, and

$$P(x) = \frac{(at)^x e^{-(at)}}{x!}$$

$P(x)$ can be approximated by a normal distribution with mean $\mu = at$ and standard deviation $\sigma = \sqrt{at}$.

## BAYESIAN LEARNING AND INFERENCING

Generally, the data given to the decision maker is obtained from past history; it is not in his control. It is not possible to design experiments either to obtain this information. Indeed many inventory design problems involving uncertainty are frequently tackled using historical or subjective data. Though somewhat bothersome for practitioners, Bayesian inferencing of such uncertainty would be specially suited when learning is expected to be taken advantage of to aid the system to improve its performance, because in such learning the existing or prior knowledge can be updated as new data is accumulated, yielding posterior inferences. Unless the situation is dynamic that must be adapted to to sustain optimality, the assumption of stationarity is generally made to keep the analysis straightforward (Graves 1999). In the present case also we assume stationarity; specifically, the parameters that control the distribution of demand are assumed to be unknown, but stationary.

Since the Bayesian learning logic (*prior* → *prior* + *data* → *posterior*) follows the path of pre-supposing information, observing the phenomena and then repeatedly updating stochastic information, it is important here to select an appropriate subjective probability function for the "prior". Indeed it is often difficult to find a suitable family of prior distributions that would help one to capture the decision maker's subjective belief, and also be algebraically workable to produce the "posterior" as new data is received. In theory this is done by deriving the posterior density from the likelihood function and the prior density, and deriving the distribution of the reduced-form parameters from the initial information on the unknown parameters controlling the stochastic process (here the random occurrence of demand).

For the present case, therefore, demand is being assumed to be random, Poisson distributed with an unknown parameter (average rate) $\lambda$/unit time. This assumption has two advantages: First, the Poisson distribution is often quite realistic when the quantities demanded are random and independent of earlier and future demands. Secondly, from analytical point of view, the Bayesian prior-posterior *conjugate family* (Raiffa and Schlaifer 1961) of the distribution of the possible values of $\lambda$ is Gamma, a two-parameter distribution that is convenient to update. However, we note that this is not a major restriction—Bayesian inference may be performed using stochastic simulation of the process also (Morales et al. 2005).

### The Bayesian Learning Framework

Poisson being the assumed distribution controlling demand with an unknown stationary rate $\lambda$/unit time, we will treat this uncertainty as a prior probability distribution, to be updated as a posterior distribution on the basis of new demand data as observed. The conjugate prior distribution for the Poisson rate parameter is the Gamma distribution with two parameters α and β, the density function being

$$Gamma(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x \geq 0, \alpha > 0, \beta > 0$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$.

In Bayesian statistics, the Gamma distribution is used as a conjugate prior (Raiffa and Schlaifer 1961) distribution for various types of rate parameters, such as the $\lambda$ of an exponential distribution or a Poisson distribution. The Gamma distribution has a mean of $\alpha/\beta$, variance $\alpha/\beta^2$ and it is known to be flexible in shape. The interpretation is as follows: There are α total occurrences in β time intervals. Updating the Gamma prior is straightforward. For instance, if $r$ quantities are demanded from the inventory in a time period of length $t$, the posterior density of $\lambda$ will be a Gamma distribution with $\alpha' = \alpha + r$ and $\beta' = \beta + t$. The maximum likelihood estimated of $\lambda$ is obtained from the posterior mean

$$E(\lambda) = \frac{\alpha + r}{\beta + t}$$

This posterior mean $E(\lambda)$ approaches $\lambda_{MLE}$ in the limit as $\alpha \to 0$ and $\beta \to 0$. In the present application our intention will be to start with some reasonably assumed prior value of Poisson demand rate λ (= α/β) and then continually update it using $r$ (the demand observed in the time span $t$) as time $t$ progresses.

The Bayesian approach to updating modeling parameters, such as the random demand that an inventory should be capable of satisfying, gives us a way around a special decision problem. In many such cases the object is to make the best decision on the basis of a given set of data (Lancaster 2004). This data may be available from past history, or it may be produced by conducting some special statistical experiments. What if there is no such history available, or there is no opportunity to conduct the desired experiments? As said above, the Bayesian approach begins with an assumed prior about the decision environment, and then by using a learning logic (*prior* → *prior* + *data* → *posterior*) follows the path of pre-supposing information, observes the phenomena and then repeatedly updates the information at hand (see Brown and Rogers 1972, Aronis et al. 2004, Morales et al. 2005).

**FIGURE 3**
**TRACE OF CONTINUALLY UPDATED POSTERIOR MEAN ESTIMATES PRODUCED BY SUCCESSION OF SAMPLES DRAWN FROM A SIMULATED POISSON DISTRIBUTION WITH $\lambda$ = 100**

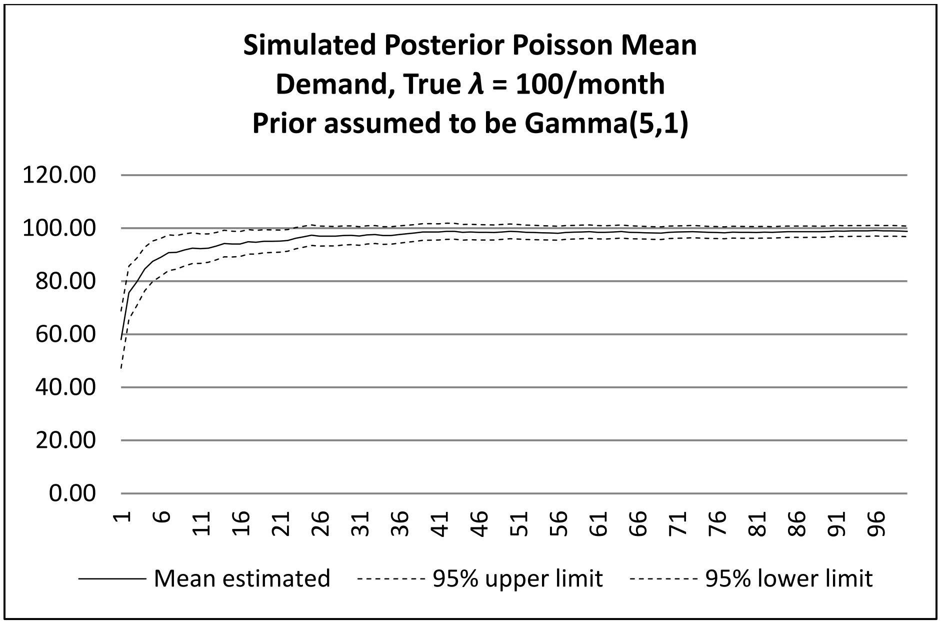



Figure 3 and Table 2 display the effect of updating the maximum likelihood estimate of the mean demand of a Poisson distribution. In this case a Poisson demand process with true mean demand ($\lambda$) of 100 units/month was observed (simulated) in succession of months 1, 2, 3, … etc. and the estimates of the posterior along with the ±95% limits of this estimate was calculated. The prior of $\lambda$ was assumed to be a Gamma($\alpha$ = 5, $\beta$ = 1) distribution with an average of 5 units/month. The simulated monthly demands are shown in the second column (ki) of Table 2. Continual updating of the estimated value of $\lambda$ produced a 95% confidence band for $\lambda$ as (96.81, 100.77) after 100 updates. Clearly, another assumed value of the prior would produce another trace of estimates. But it can be assured based on the standard deviation of the estimated $\lambda$ that they would all eventually converge toward the true demand as time *t* increases.

## REGULAR BAYESIAN UPDATES OF DEMAND RATE HELP KEEP TOTAL EXPECTED COST NEAR MINIMUM

It is clear from the foregoing section that when demand is unknown and stochastic, the original assumption about demand shifts gradually toward the true demand when the estimate of the true demand is suitably updated. One method for accomplishing this would be to incorporate Bayesian learning. This section adds in Bayesian learning into the (*s, Q*) policy to help evaluate the utility of doing this. Such learning would help in moving the operating values of the two decision parameters—*s*, the reorder point, and *Q*, the order quantity—closer to their optimal values as time advances and new demand data become known. For effectively using the (*s, Q*) policy the motivation for continually adjusting *s* and *Q* therefore would be to keep the total expected inventory management cost near its minimum throughout the period for which the inventory is maintained and managed.

**TABLE 2**
**100 POISSON RANDOM VALUES SIMULATED WITH $\lambda$ = 100. ASSUMING A PRIOR AS A GAMMA($\alpha$ = 5, $\beta$ = 1) DISTRIBUTION, WE COMPUTE MEAN OF THE GAMMA POSTERIOR DISTRIBUTIONS AS MONTHLY DEMAND IS SUCCESSIVELY OBSERVED**

| Month # | Observed Monthly Demand | | For Posterior Gamma ($\underline{\alpha}$, $\underline{\beta}$): | | $\lambda$ = Mean of Gamma Posterior | SD = SQRT($\underline{\alpha}/\underline{\beta}^2$) | Upper/Lower estimates of $\lambda$ | |
|---|---|---|---|---|---|---|---|---|
| i | ki | Sum ki | $\underline{\alpha} = \alpha$ + sum ki | $\underline{\beta} = \beta + n$ | $\lambda$ estimate = $\underline{\alpha}/\underline{\beta}$ | SD ($\lambda$ est) | +95% | -95% |
| Month 1 | 111 | 111 | 116 | 2 | 58.00 | 5.39 | 68.77 | 47.23 |
| 2 | 111 | 222 | 227 | 3 | 75.67 | 5.02 | 85.71 | 65.62 |
| 3 | 92 | 314 | 319 | 4 | 79.75 | 4.47 | 88.68 | 70.82 |
| 4 | 104 | 418 | 423 | 5 | 84.60 | 4.11 | 92.83 | 76.37 |
| 5 | 102 | 520 | 525 | 6 | 87.50 | 3.82 | 95.14 | 79.86 |
| 6 | 98 | 618 | 623 | 7 | 89.00 | 3.57 | 96.13 | 81.87 |
| 7 | 103 | 721 | 726 | 8 | 90.75 | 3.37 | 97.49 | 84.01 |
| 8 | 92 | 813 | 818 | 9 | 90.89 | 3.18 | 97.24 | 84.53 |
| 9 | 100 | 913 | 918 | 10 | 91.80 | 3.03 | 97.86 | 85.74 |
| 10 | 99 | 1012 | 1017 | 11 | 92.45 | 2.90 | 98.25 | 86.66 |
| 11 | 90 | 1102 | 1107 | 12 | 92.25 | 2.77 | 97.80 | 86.70 |
| 12 | 95 | 1197 | 1202 | 13 | 92.40 | 2.67 | 97.80 | 87.13 |
| 13 | 103 | 1300 | 1305 | 14 | 93.21 | 2.58 | 98.37 | 88.05 |
| 14 | 108 | 1408 | 1413 | 15 | 94.20 | 2.51 | 99.21 | 89.19 |
| 15 | 91 | 1499 | 1504 | 16 | 94.00 | 2.42 | 98.85 | 89.15 |
| 16 | 94 | 1593 | 1598 | 17 | 94.00 | 2.35 | 98.70 | 89.30 |
| 17 | 109 | 1702 | 1707 | 18 | 94.83 | 2.30 | 99.42 | 90.24 |
| 18 | 92 | 1794 | 1799 | 19 | 94.68 | 2.23 | 99.15 | 90.22 |
| 19 | 102 | 1896 | 1901 | 20 | 95.05 | 2.18 | 99.41 | 90.69 |
| 20 | 95 | 1991 | 1996 | 21 | 95.05 | 2.13 | 99.30 | 90.79 |
| 21 | 96 | 2087 | 2092 | 22 | 95.09 | 2.08 | 99.25 | 90.93 |
| 22 | 101 | 2188 | 2193 | 23 | 95.35 | 2.04 | 99.42 | 91.28 |
| 23 | 116 | 2304 | 2309 | 24 | 96.21 | 2.00 | 100.21 | 92.20 |
| 24 | 110 | 2414 | 2419 | 25 | 96.76 | 1.97 | 100.69 | 92.83 |
| 25 | 112 | 2526 | 2531 | 26 | 97.35 | 1.93 | 101.22 | 93.48 |
| 26 | 88 | 2614 | 2619 | 27 | 97.00 | 1.90 | 100.79 | 93.21 |
| 27 | 97 | 2711 | 2716 | 28 | 97.00 | 1.86 | 100.72 | 93.28 |
| 28 | 96 | 2807 | 2812 | 29 | 96.97 | 1.83 | 100.62 | 93.31 |
| 29 | 105 | 2912 | 2917 | 30 | 97.23 | 1.80 | 100.83 | 93.63 |
| 30 | 99 | 3011 | 3016 | 31 | 97.29 | 1.77 | 100.83 | 93.75 |
| 31 | 89 | 3100 | 3105 | 32 | 97.03 | 1.74 | 100.51 | 93.55 |
| 32 | 111 | 3211 | 3216 | 33 | 97.45 | 1.72 | 100.89 | 94.02 |
| 33 | 102 | 3313 | 3318 | 34 | 97.59 | 1.69 | 100.98 | 94.20 |
| 34 | 85 | 3398 | 3403 | 35 | 97.23 | 1.67 | 100.56 | 93.90 |
| 35 | 97 | 3495 | 3500 | 36 | 97.22 | 1.64 | 100.51 | 93.94 |
| 36 | 111 | 3606 | 3611 | 37 | 97.59 | 1.62 | 100.84 | 94.35 |
| 37 | 109 | 3715 | 3720 | 38 | 97.89 | 1.61 | 101.10 | 94.68 |
| 38 | 108 | 3823 | 3828 | 39 | 98.15 | 1.59 | 101.33 | 94.98 |
| 39 | 113 | 3936 | 3941 | 40 | 98.53 | 1.57 | 101.66 | 95.39 |
| 40 | 100 | 4036 | 4041 | 41 | 98.56 | 1.55 | 101.66 | 95.46 |
| 41 | 99 | 4135 | 4140 | 42 | 98.57 | 1.53 | 101.64 | 95.51 |
| 42 | 109 | 4244 | 4249 | 43 | 98.81 | 1.52 | 101.85 | 95.78 |
| 43 | 98 | 4342 | 4347 | 44 | 98.80 | 1.50 | 101.79 | 95.80 |
| 44 | 83 | 4425 | 4430 | 45 | 98.44 | 1.48 | 101.40 | 95.49 |
| 45 | 103 | 4528 | 4533 | 46 | 98.54 | 1.46 | 101.47 | 95.62 |
| 46 | 92 | 4620 | 4625 | 47 | 98.40 | 1.45 | 101.30 | 95.51 |
| 47 | 98 | 4718 | 4723 | 48 | 98.40 | 1.43 | 101.26 | 95.53 |
| 48 | 96 | 4814 | 4819 | 49 | 98.35 | 1.42 | 101.18 | 95.51 |
| 49 | 109 | 4923 | 4928 | 50 | 98.56 | 1.40 | 101.37 | 95.75 |
| 50 | 109 | 5032 | 5037 | 51 | 98.76 | 1.39 | 101.55 | 95.98 |

**TABLE 2 (CONTD.)**
**100 POISSON RANDOM VALUES SIMULATED WITH λ = 100. ASSUMING A PRIOR AS A GAMMA(α = 5, β = 1) DISTRIBUTION, WE COMPUTE MEAN OF THE GAMMA POSTERIOR DISTRIBUTIONS AS MONTHLY DEMAND IS SUCCESSIVELY OBSERVED**

| Month # | Observed Monthly Demand | | For Posterior Gamma ($\underline{\alpha}$, $\underline{\beta}$): | | $\lambda$ = Mean of Gamma Posterior | SD = SQRT($\underline{\alpha}/\underline{\beta}^2$) | Upper/Lower estimates of $\lambda$ | |
|---|---|---|---|---|---|---|---|---|
| i | ki | Sum ki | $\underline{\alpha} = \alpha + \text{sum ki}$ | $\underline{\beta} = \beta + n$ | $\lambda$ estimate = $\underline{\alpha}/\underline{\beta}$ | SD ($\lambda$ est) | +95% | -95% |
| 51 | 93 | 5125 | 5130 | 52 | 98.65 | 1.38 | 101.41 | 95.90 |
| 52 | 86 | 5211 | 5216 | 53 | 98.42 | 1.36 | 101.14 | 95.69 |
| 53 | 96 | 5307 | 5312 | 54 | 98.37 | 1.35 | 101.07 | 95.67 |
| 54 | 92 | 5399 | 5404 | 55 | 98.25 | 1.34 | 100.93 | 95.58 |
| 55 | 93 | 5492 | 5497 | 56 | 98.16 | 1.32 | 100.81 | 95.51 |
| 56 | 93 | 5585 | 5590 | 57 | 98.07 | 1.31 | 100.69 | 95.45 |
| 57 | 116 | 5701 | 5706 | 58 | 98.38 | 1.30 | 100.98 | 95.77 |
| 58 | 103 | 5804 | 5809 | 59 | 98.46 | 1.29 | 101.04 | 95.87 |
| 59 | 105 | 5909 | 5914 | 60 | 98.57 | 1.28 | 101.13 | 96.00 |
| 60 | 105 | 6014 | 6019 | 61 | 98.67 | 1.27 | 101.22 | 96.13 |
| 61 | 85 | 6099 | 6104 | 62 | 98.45 | 1.26 | 100.97 | 95.93 |
| 62 | 95 | 6194 | 6199 | 63 | 98.40 | 1.25 | 100.90 | 95.90 |
| 63 | 109 | 6303 | 6308 | 64 | 98.56 | 1.24 | 101.04 | 96.08 |
| 64 | 108 | 6411 | 6416 | 65 | 98.71 | 1.23 | 101.17 | 96.24 |
| 65 | 81 | 6492 | 6497 | 66 | 98.44 | 1.22 | 100.88 | 96.00 |
| 66 | 94 | 6586 | 6591 | 67 | 98.37 | 1.21 | 100.80 | 95.95 |
| 67 | 91 | 6677 | 6682 | 68 | 98.26 | 1.20 | 100.67 | 95.86 |
| 68 | 92 | 6769 | 6774 | 69 | 98.17 | 1.19 | 100.56 | 95.79 |
| 69 | 93 | 6862 | 6867 | 70 | 98.10 | 1.18 | 100.47 | 95.73 |
| 70 | 120 | 6982 | 6987 | 71 | 98.41 | 1.18 | 100.76 | 96.05 |
| 71 | 107 | 7089 | 7094 | 72 | 98.53 | 1.17 | 100.87 | 96.19 |
| 72 | 104 | 7193 | 7198 | 73 | 98.60 | 1.16 | 100.93 | 96.28 |
| 73 | 105 | 7298 | 7303 | 74 | 98.69 | 1.15 | 101.00 | 96.38 |
| 74 | 87 | 7385 | 7390 | 75 | 98.53 | 1.15 | 100.83 | 96.24 |
| 75 | 89 | 7474 | 7479 | 76 | 98.41 | 1.14 | 100.68 | 96.13 |
| 76 | 94 | 7568 | 7573 | 77 | 98.35 | 1.13 | 100.61 | 96.09 |
| 77 | 91 | 7659 | 7664 | 78 | 98.26 | 1.12 | 100.50 | 96.01 |
| 78 | 115 | 7774 | 7779 | 79 | 98.47 | 1.12 | 100.70 | 96.24 |
| 79 | 95 | 7869 | 7874 | 80 | 98.43 | 1.11 | 100.64 | 96.21 |
| 80 | 98 | 7967 | 7972 | 81 | 98.42 | 1.10 | 100.62 | 96.22 |
| 81 | 96 | 8063 | 8068 | 82 | 98.39 | 1.10 | 100.58 | 96.20 |
| 82 | 104 | 8167 | 8172 | 83 | 98.46 | 1.09 | 100.64 | 96.28 |
| 83 | 93 | 8260 | 8265 | 84 | 98.39 | 1.08 | 100.56 | 96.23 |
| 84 | 112 | 8372 | 8377 | 85 | 98.55 | 1.08 | 100.71 | 96.40 |
| 85 | 106 | 8478 | 8483 | 86 | 98.64 | 1.07 | 100.78 | 96.50 |
| 86 | 101 | 8579 | 8584 | 87 | 98.67 | 1.06 | 100.80 | 96.54 |
| 87 | 96 | 8675 | 8680 | 88 | 98.64 | 1.06 | 100.75 | 96.52 |
| 88 | 100 | 8775 | 8780 | 89 | 98.65 | 1.05 | 100.76 | 96.55 |
| 89 | 98 | 8873 | 8878 | 90 | 98.64 | 1.05 | 100.74 | 96.55 |
| 90 | 105 | 8978 | 8983 | 91 | 98.71 | 1.04 | 100.80 | 96.63 |
| 91 | 119 | 9097 | 9102 | 92 | 98.93 | 1.04 | 101.01 | 96.86 |
| 92 | 92 | 9189 | 9194 | 93 | 98.86 | 1.03 | 100.92 | 96.80 |
| 93 | 106 | 9295 | 9300 | 94 | 98.94 | 1.03 | 100.99 | 96.88 |
| 94 | 99 | 9394 | 9399 | 95 | 98.94 | 1.02 | 100.98 | 96.90 |
| 95 | 99 | 9493 | 9498 | 96 | 98.94 | 1.02 | 100.97 | 96.91 |
| 96 | 111 | 9604 | 9609 | 97 | 99.06 | 1.01 | 101.08 | 97.04 |
| 97 | 88 | 9692 | 9697 | 98 | 98.95 | 1.00 | 100.96 | 96.94 |
| 98 | 103 | 9795 | 9800 | 99 | 98.99 | 1.00 | 100.99 | 96.99 |
| 99 | 88 | 9883 | 9888 | 100 | 98.88 | 0.99 | 100.87 | 96.89 |
| 100 | 90 | 9973 | 9978 | 101 | 98.79 | 0.99 | 100.77 | 96.81 |

One way to operate the (*s, Q*) inventory policy when demand is stochastic is to be myopic (Levy et al. 2007) when one sets the operating values of *s* and *Q* at some initial (prior) guess for the demand rate *λ*, or perhaps collects only a limited amount of demand data and then estimates *λ*. Such trust on an initial guess for *λ* and not changing it later may even be favored, for this may save the added effort needed to incorporate any emergent evidence about true demand as the business moves forward and one becomes busy. Many practitioners indeed do not change the initial assumption about *a* or *λ* or even costs, though Azoury and Miller (1994) have amply highlighted the effect of not utilizing emerging information—by comparing Bayesian and non-Bayesian methodologies for it.

Plainly, the *s* and *Q* derived using the initial guess for *λ* would almost surely be suboptimal, except by accident (Figure 2 indicates the strong dependence of EC on *a* (hence on *λ*)). To test any possible merit of the myopic approach we set the course computationally. Table 3 displays the effect of setting the unknown demand rate *λ* at some unsubstantiated value (assuming mistakenly this to be the true demand), deriving the flawed $s^*$ and $Q^*$ from it, and incurring the consequent expected total cost EC when the (*s, Q*) policy is used. It is straightforward to compare these higher values of EC with the near optimal EC achievable by getting close to the *true demand rate*. We show this here by using the Bayesian approach. The illustration in Table 3 used 100 units/month as the true demand rate whereas the ("wrongly") presumed values of *λ* were set respectively at 25, 50, 75, 150, 200, and 300 units/month. The costs used were *h* = \$10/unit-month, π1 = \$500 per backorder event and *K* = \$800/order placed.

Figures 4 and 5 respectively display the effect of using a *flawed* (differing from the true) demand conjecture ("prior") on the expected total cost (EC), and the service level experienced by customers.

**TABLE 3**
**COSTS AND SERVICE LEVELS EXPERIENCED WHEN $s^*$ AND $Q^*$ ARE SET BASED ON A PRIOR ESTIMATED DEMAND, BUT TRUE DEMAND (*aT*) IS DIFFERENT FROM THE PRIOR ESTIMATE (*a*)**

| ***a*** **= Estimated demand/month based on a *flawed* prior guess** | **25** | **50** | **75** | **100** | **150** | **200** | **300** |
|---|---|---|---|---|---|---|---|
| ***aT*** **= True demand/month** | **100** | **100** | **100** | **100** | **100** | **100** | **100** |
| ***s**** **= Optimum Reorder Point based on prior** | 10 | 18 | 25 | 33 | 47 | 61 | 88 |
| ***Q**** **= Optimum Order Quantity based on prior** | **65** | **91** | **111** | **129** | **158** | **183** | **224** |
| **Holding Cost based on *a and (s*, Q*)*** | 360 | 510 | 624 | 720 | 883 | 1019 | 1248 |
| **Holding Cost based on *(s*, Q*) but facing True demand*** | **173** | **385** | **569** | **721** | **989** | **1269** | **1748** |
| **Replenishment Cost by *a and (s*, Q*)*** | 310 | 438 | 537 | 620 | 759 | 876 | 1073 |
| **Replenishment Cost based on *(s*, Q*) but facing True demand*** | **1239** | **876** | **715** | **620** | **506** | **438** | **358** |
| **Shortage Cost by *a and (s*, Q*)*** | 13 | 18 | 22 | 26 | 32 | 36 | 45 |
| **Shortage Cost based on *(s*, Q*) but facing True demand*** | **52** | **36** | **21** | **26** | **36** | **18** | **15** |
| **Expected Total Cost by *a and (s*, Q*)*** | 683 | 966 | 1183 | 1366 | 1673 | 1932 | 2366 |
| **Expected Total Cost based on *(s*, Q*) but facing True demand*** | **1464** | **1297** | **1305** | **1366** | **1531** | **1725** | **2121** |
| **Service level experienced at True Demand but operating at (*s*, Q**)** | 98.8 % | 97.2 % | 95.4 % | 93.4 % | 88.5 % | 82.5 % | 59.6 % |

**FIGURE 4**
**THE EXPECTED TOTAL COST FOR A (*s*, *Q*) SYSTEM THAT USES A FLAWED PRIOR ESTIMATE THAT IS FAR FROM THE TRUE DEMAND VALUE**

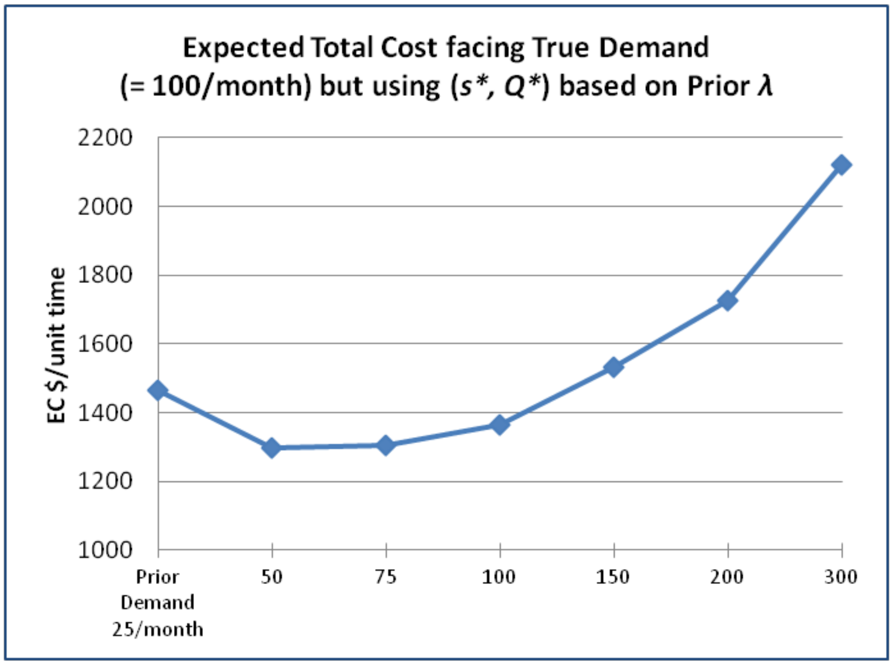
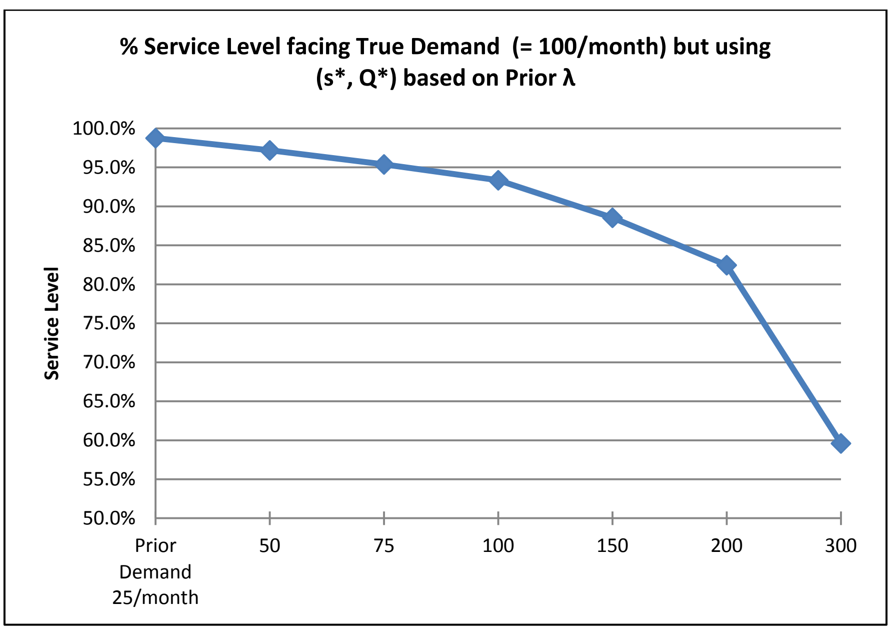



**FIGURE 5**
**SERVICE LEVEL PROVIDED BY A (*s*, *Q*) SYSTEM THAT USES A FLAWED PRIOR ESTIMATE THAT IS FAR FROM THE TRUE DEMAND VALUE**



A review of Table 3 and Figures 4 and 5 would suggest that the results of using Bayesian learning to keep continually updating the demand estimate provide a mixed message. But a closer look reveals that the minimum total cost ($s$, $Q$) policy should indeed be based on a demand estimate *as close to* the true demand as is possible. But Figure 4 appears to suggest that a lower guess for demand actually improves customer service! Prima facie, therefore, guessing a low value of demand appears to be doing something good. But such inference is shortsighted and most misleading. More seriously, this is not a defect in the *model* or its analysis, rather perhaps in the analyst's formulation of the problem at hand.

Recall that our objective of setting up the ($s$, $Q$) model to help find a rational way to manage inventories when demand is stochastic included spelling out the objective first—that of minimizing (1), the expected total cost/unit time. This total cost included three components—the holding cost, the replenishment cost, and the shortage cost. At least for this model, therefore, maximizing customer service per se was not the objective. Customer service ($F(s)$) enters into (1) via $Cs$ ($= \pi 1(1 - F(s))$), the cost of short shipment. If one requires the final ($s$, $Q$) solution to assure a high level of customer service, one would need to use a large value for $\pi 1$. Thus, like any optimization attempted, one would need to be clear about the objective of the decision maker—where does he want to put priority?

## CONCLUSIONS

This study has investigated the value of incorporating Bayesian learning into the popular ($s$, $Q$) model for managing inventories when demand is stochastic. The study finds that one can indeed make a decent start by the Bayesian approach—*and stay the course nearly optimally*—while upholding a target service level by suitably selecting costs *and also* keeping the expected operating cost/unit time minimum. Specifically, this study uncovers the high value in continually updating the decisions (a) "when to order", and (b) "how much to order", rather than sticking to the initial guess for the demand average, as is frequently practiced.

Demand has been assumed to follow a stationary Poisson distribution in this work. The Bayesian learning process uses the Gamma conjugate family of distributions to incorporate the latest observed demand data. The learning logic follows the path *prior* → *prior* + *data* → *posterior* of observing and updating stochastic information.

Owing to the non-linear and complex nature of the expressions in the ($s$, $Q$) model, this work utilized an orthogonal array experimental framework to determine the sensitivity of the two performances (responses)--% service level and total expected cost. For this two working levels for each of the factors—monthly demand ($a$), holding cost ($h$), shortage cost ($\pi 1$) and order cost ($K$) were selected and a $L_8$ array was adopted to guide the computations. The following inferences could be drawn:

- Optimal Order Quantity $Q^*$ is affected significantly by $a$, $h$ and $K$, but only mildly by shortage cost $\pi 1$.
- Service level seems to be robust relative to most factors considered in the region of the cost parameters studied. It is closely related to shortage cost $\pi 1$ and the setting of the reorder point $s^*$.
- Total Expected Cost/unit time is relatively robust with respect to shortage cost per stock out event $\pi 1$, *but* sensitive to $a$, $h$ and $K$.
- A myopic estimate of the true demand rate $\lambda$ would regularly turn out a suboptimal EC.

These deductions—typically unavailable to the inventory manager—suggest that it would be wise to spend effort in *optimally setting* the reorder point $s^*$ and order quantity $Q^*$ before one sets out to declare the operational policies to cope with stochastic demand. In a sense, this information is comparable to the knowledge of *relative robustness* of the total operating cost/unit time to EOQ for a deterministic inventory system (see Winston 2003, page 853).

In summary, this study, like the work of Azoury and Miller (1994) conducted for the $n$-period nondepletive inventory model, establishes the worth of following the Bayesian approach to update stochastic information used in the ($s$, $Q$) inventory system as new demand data become continually available. The result is lowering of the total expected cost throughout the time this inventory system is operated. As already shown in the literature in a large number of applications, the Bayesian learning path *prior* → *prior + data* → *posterior*, it appears, would benefit the design and operation of other similar stochastic systems as well.

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